Temporal Graph Mining

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Tutors

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Website

https://miningtemporalnetworks.github.io/

Interaction

`menti.com`

code: 14 46 97 6

Agenda

Part I : **Introduction and Motivation**

- models of temporal networks
- algorithmic approaches

Part II : **Mining Temporal Networks A:**

- connectivity, temporal properties
- centrality, cores

Part III : **Mining Temporal Networks B:**

- communities, patterns and events
- diffusion and random networks

Part IV : **Tools and Code Libraries**

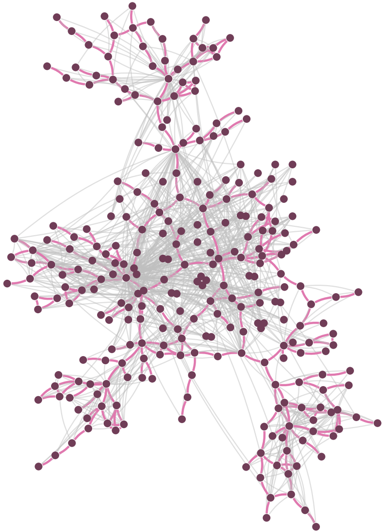
Part V : **Challenges, Open Problems, and Trends**

Part I

Introduction and Motivation

Interconnected World

- networks model objects and their relations



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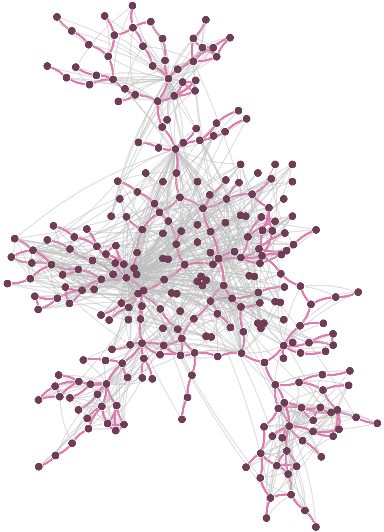

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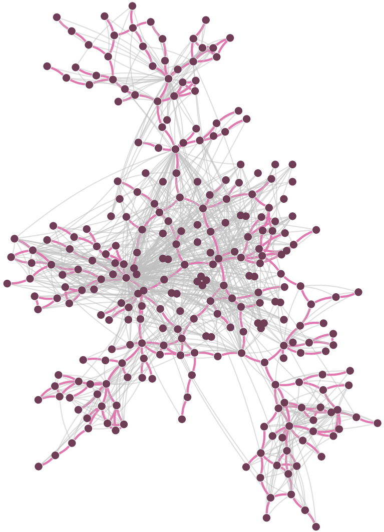

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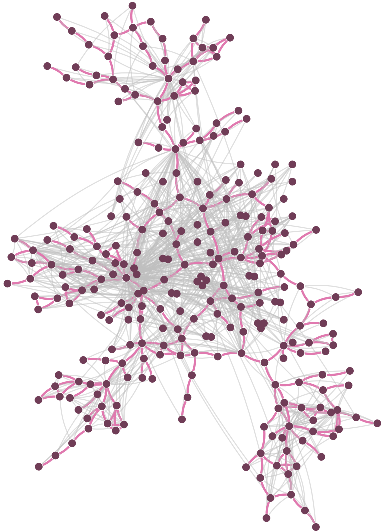

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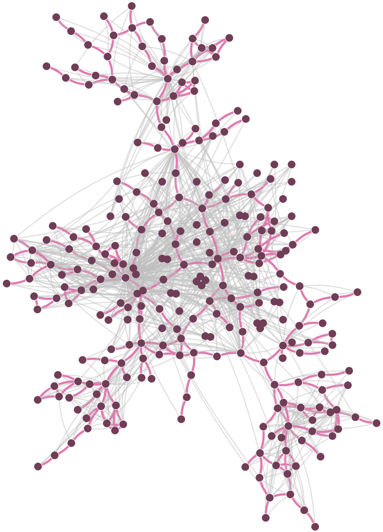

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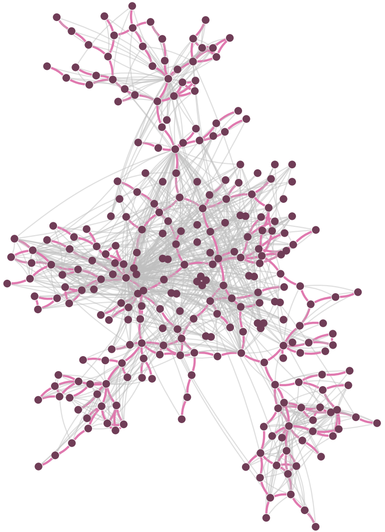

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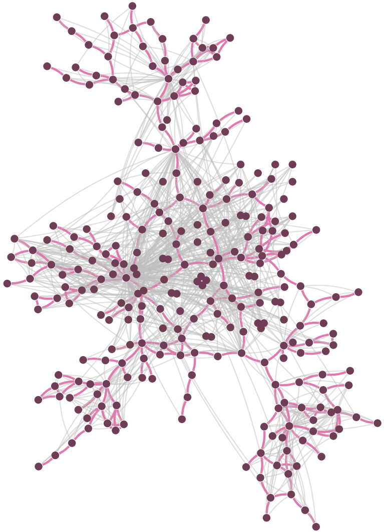
- networks are ubiquitous in WWW-based applications



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- online communication networks and social media

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- insights can lead to huge monetary and societal impacts

Research Questions in Network Mining

- structure discovery
  - communities, summarization, events, role mining

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- study complex dynamic phenomena
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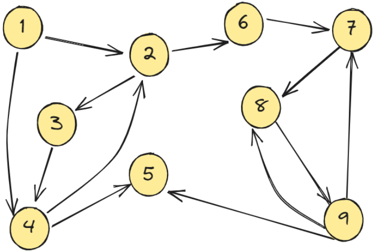
- design efficient algorithms

Network Mining: Traditional View

- networks represented as pure graph-theory objects

  no additional vertex / edge information



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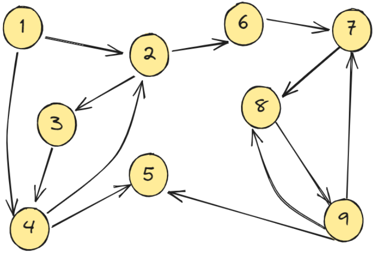
- emphasis on static networks



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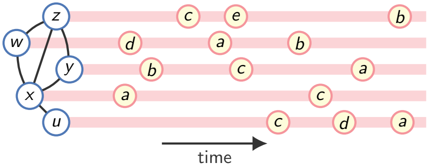
- capturing activity and interaction occurring over systems
- giving rise to new concepts, new problems, and new computational challenges and opportunities

Modeling Activity in Networks

1. network nodes perform actions (e.g., posting messages)



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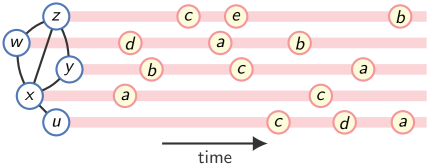
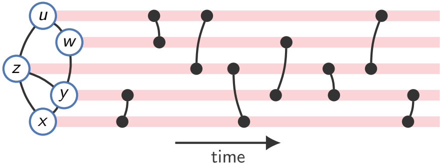
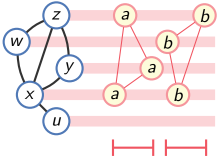


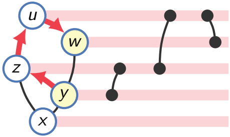
2. network nodes interact with each other
   (e.g., a "like", a repost, or sending a message to each other)



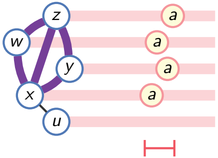
Many Novel and Interesting Concepts



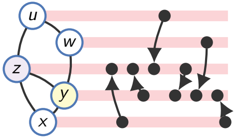
new pattern types



temporal information paths



new types of events



network evolution

Temporal Network Mining — Objectives

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- develop algorithmic approaches
- analyze real-world data and gain novel insights

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Y:

- networks
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- some combinations have distinct meaning, but not always

Examples of Temporal Networks

- online communication networks
  - phone, email, text messages, etc.

(Holme, 2015)

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- bibliographic networks
  - collaboration and citation networks

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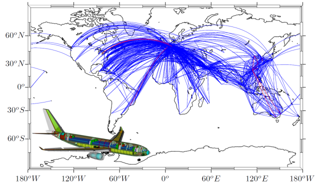
Examples of Temporal Networks

- human proximity networks
  - recorded by various sensors and devices, e.g., bluetooth, wifi, etc.
  - patient-referral networks, i.e., how patients are transferred between wards of a hospital system
  - sexual contact networks



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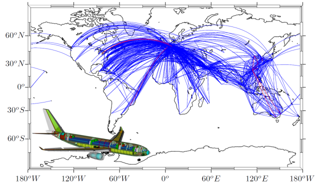
- travel and transportation networks
  - airline connections, bus transport, bike-sharing systems



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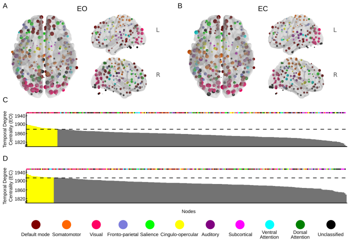

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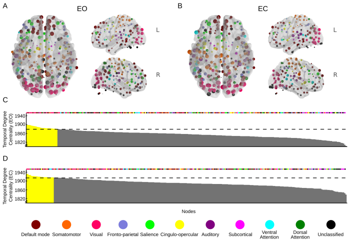

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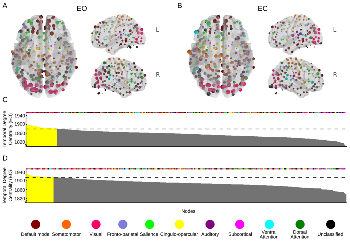
- animal proximity networks
  - obtained via RFID devices
  - lifestock or wildlife



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- this is a lossless representation
- also known as sequence of contacts, or sequence of (temporal) edges or temporal edge stream
- usually, edges given in chronological order

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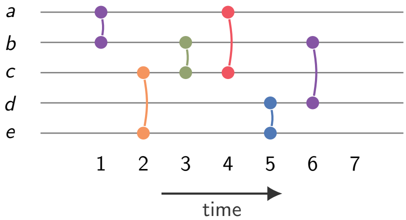
1. Sequence of interactions

- visual representation of a temporal network as a sequence of interactions



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- not equivalent representation with sequence of interactions
  - representation depends on quantization parameter, e.g., seconds, minutes, hours, days, etc.

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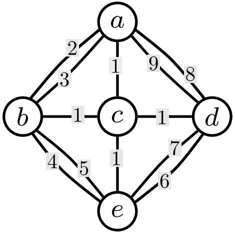
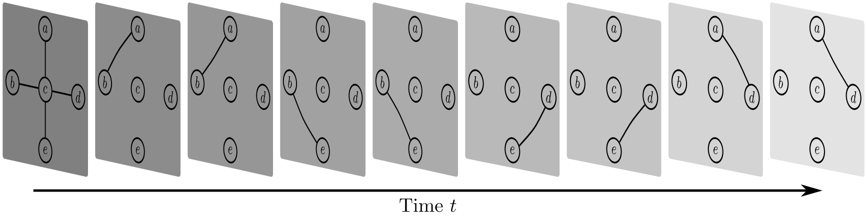
- visual representation of a temporal network as a sequence of static graphs





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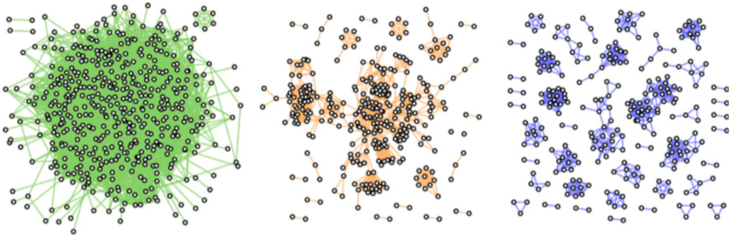
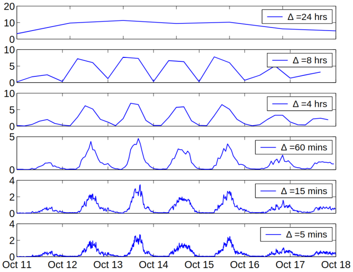


time point in contact network with time resolution of 24h, 1h, and 5 minutes (Lehmann, 2019)



mean degrees for different time resolutions (Clauset and Eagle, 2012)

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3. Time series of contacts

- a time-series for each pair of nodes in the network
- equivalent representation with sequence of interactions

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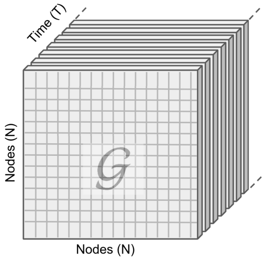
4. Tensor representation

- tensor representing node $\times$ node $\times$ time information
- can apply powerful tensor-algebra techniques
- a complication is that time is directed, while tensor algebra assumes that indices can be relabeled (breaking the time ordering)
- equivalent representation with sequence of interactions



Representation of Temporal Networks

(Casteigts et al., 2012)

5. Time-varying graphs defined as $G = (V, E, T, p, \lambda)$, where
   - V : set of nodes
   - $E \subseteq V \times V$: set of edges
   - T : a time domain
   - $p : E \times T \to \{0, 1\}$: a presence function
   - $\lambda : E \times T \to \mathbb{R}$: a latency function

- equivalent representation with sequence of interactions

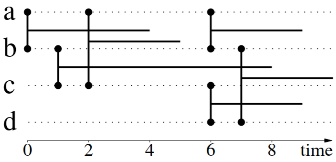
Representation of Temporal Networks

(Latapy et al., 2018)

6. Stream graphs and link streams

- a formalization for modeling interactions over time
- a stream graph is defined as $G = (T, V, W, E)$, where
  - T : a time domain
  - V : a set of nodes
  - $W \subseteq T \times V$: a set of temporal nodes
  - $E \subseteq T \times V \times V$: a set of links

    s.t., $(t, u, v) \in E$ implies $(t, u) \in W$ and $(t, v) \in W$
- stream graph: nodes are temporal too



- link stream equivalent representation with sequence of interactions

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(Holme, 2015)

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  - and with $V_I = \{u, v \mid (u, v) \in E_I\}$

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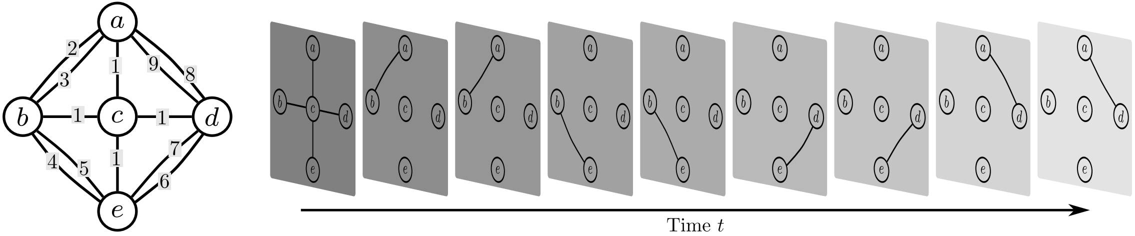
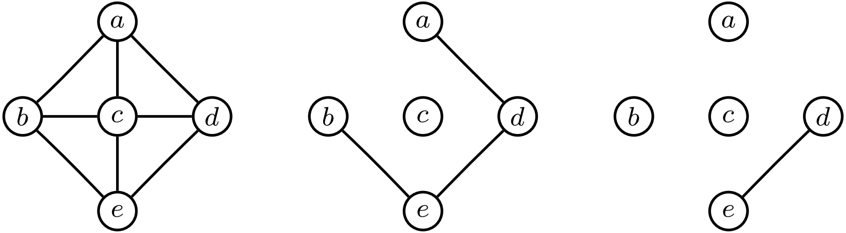
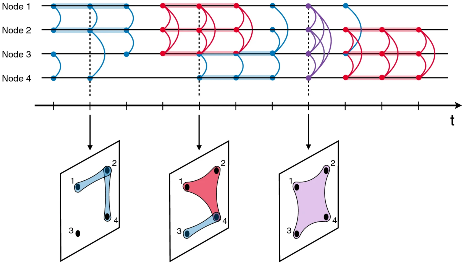




time window graphs for intervals $[1, 9]$, $[4, 9]$, $[6, 7]$

Temporal Graph Variants

- time-intervals instead of time stamps
- directed vs. undirected edges
- multi edges
- (time-variant) labeled or colored nodes and edges
- (time-variant) node and edge features
- temporal hypergraphs



(Cencetti et al., 2021)

Combinations possible: temporal multi-layer hypergraphs with node features

Temporal Networks vs. Dynamic Graphs

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  - e.g., cost of maintaining a minimum spanning tree per edge additions/deletions

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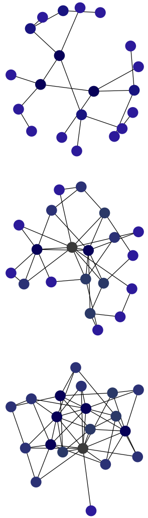

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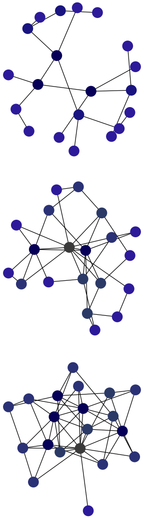

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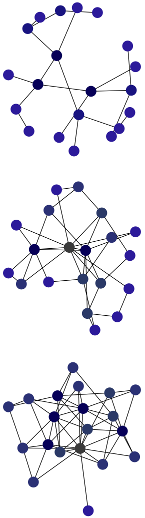

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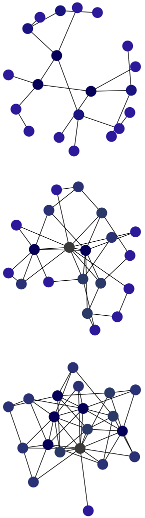

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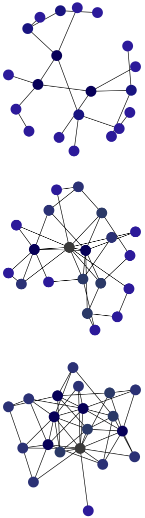

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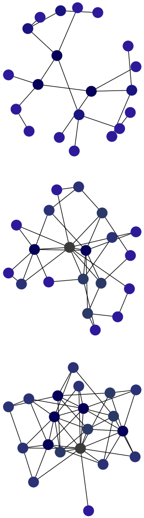

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- we can study questions of mining temporal networks in the graph-stream model

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- a number of recent works
  - graph coloring (Mertzios et al., 2018)
  - maximal matching (Mertzios et al., 2019)
  - cliques (Viard et al., 2015, 2016; Himmel et al., 2017; Mertzios et al., 2024)
  - network design (Akrida et al., 2017; Enright et al., 2021)
  - path problems (Casteigts et al., 2021; Klobas et al., 2022, 2023)
  - vertex cover (Hamm et al., 2022; Akrida et al., 2020)
  - ...
- discussing complexity, FPT algorithms, enumeration, etc.

Agenda

Part I : **Introduction and Motivation**

- models of temporal networks
- algorithmic approaches

Part II : **Mining Temporal Networks A:**

- connectivity, temporal properties
- centrality, cores

Part III : **Mining Temporal Networks B:**

- communities, patterns and events
- diffusion and random networks

Part IV : **Tools and Code Libraries**

Part V : **Challenges, Open Problems, and Trends**

Part II

Mining Temporal Networks A

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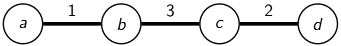
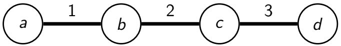
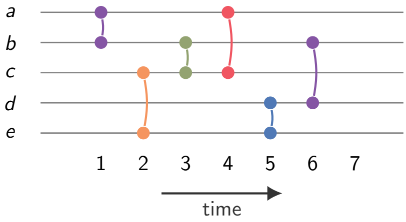




Time-respecting Paths — Example



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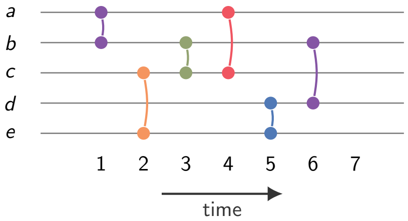

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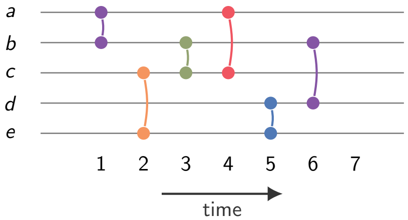

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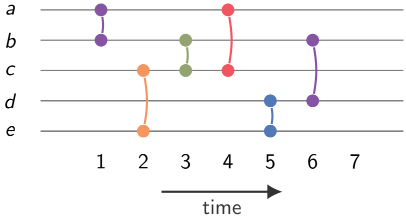

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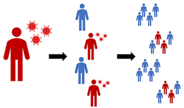
- information (or disease) can only propagate over time-respecting walks



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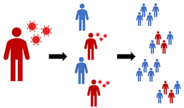

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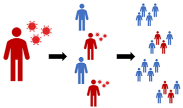

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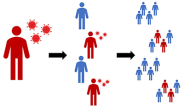

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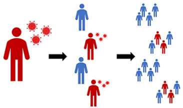

- social network analysis: centrality measures for ranking users

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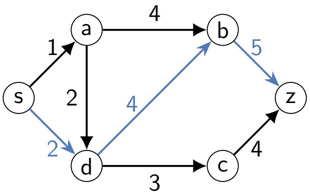

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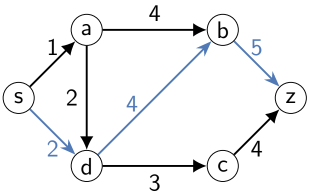

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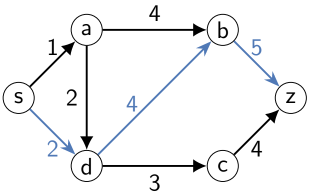

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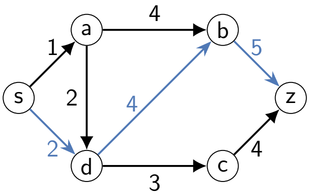

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- finding all from s reachable nodes in linear time (later)

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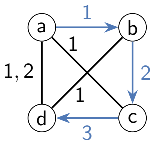

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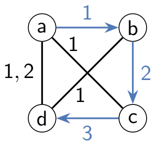

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Proof idea:

- reduction from Hamilton path problem

(Michail and Spirakis, 2016)

Connectivity and Connected Components

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  - each node in connected component C can reach each all other nodes in C
- equivalence relation that partitions the graph
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  - reflexivity, symmetry, transitivity

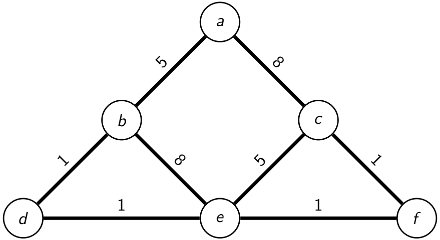
- temporal connected components are based on temporal reachability
  - a subset of the nodes $C \subseteq V$
  - there is a temporal walk between each pair $u, v \in C$

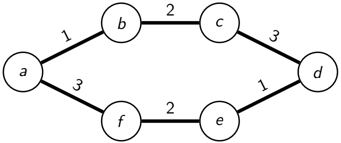
Connectivity and Connected Components

Temporal Connectivity Problem

- **Given:** A temporal graph and integer k
- **Question:** Is there a subset of the vertices $V' \subseteq V$ of size k such that all vertices in V' can reach each other by a temporal path?

- two versions: open variant allows paths using nodes outsides of V', closed variant not
- both cases are NP-complete (Bhadra and Ferreira, 2003)





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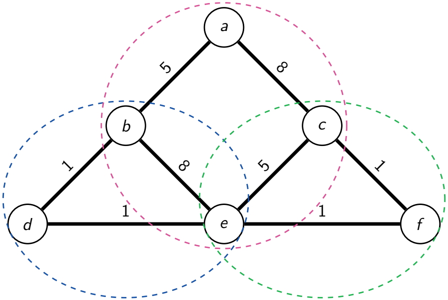

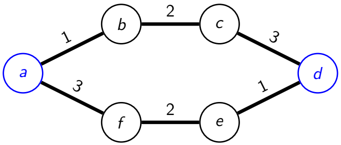

a and d are openly connected

Connected components can be overlapping

Time Window Reachability and Connectivity

- compute reachability or connected components in time window graph $G_I = (V_I, E_I)$
  - given time window $I = [a, b]$, $G_I = (V_I, E_I)$ is static graph induced by edges appearing in I

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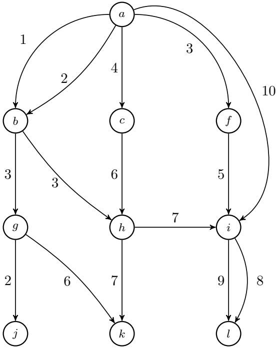
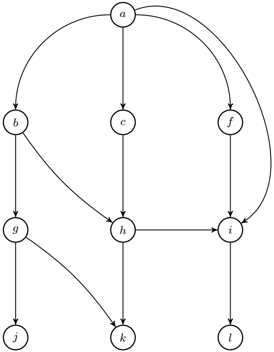
  - the set of static connected components in $G_I = (V_I, E_I)$
- both problems solvable in linear time
- alternative: index-based algorithms for large scale graphs

Time-Respecting Paths



(a) Temporal Graph



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- some paths in the static graph are not meaningful in the temporal graph

(Wu et al., 2014)

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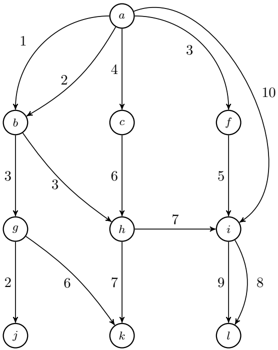

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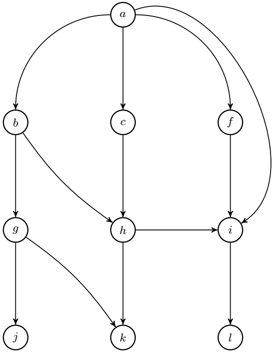

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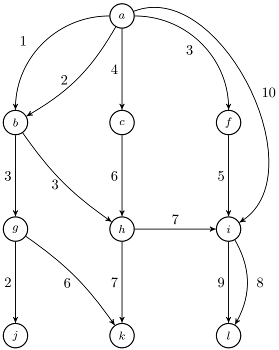

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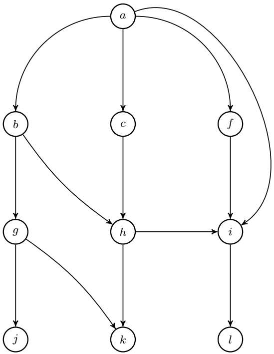

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Minimum Temporal Paths

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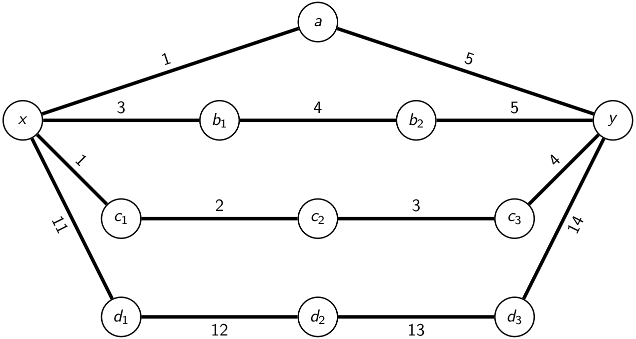

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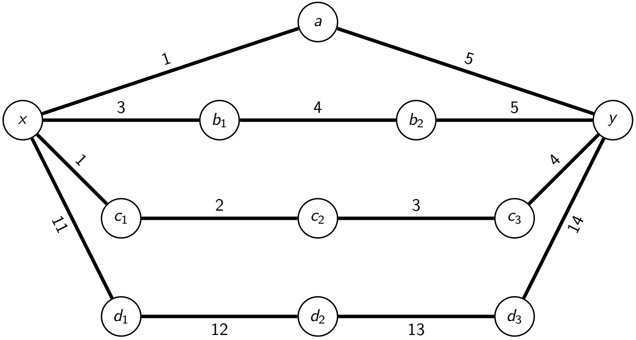

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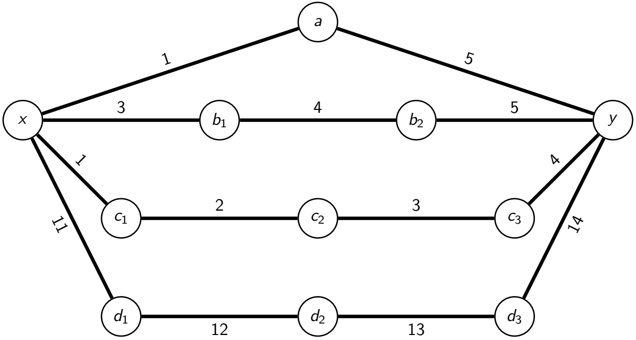

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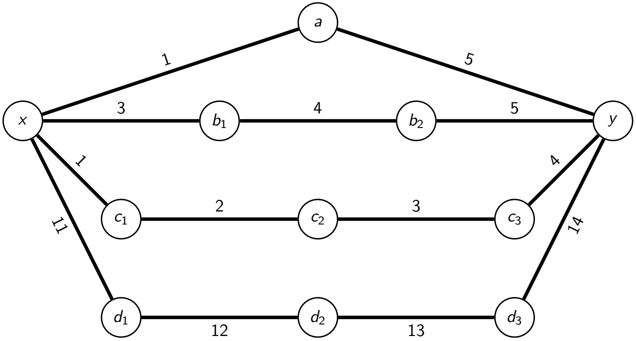

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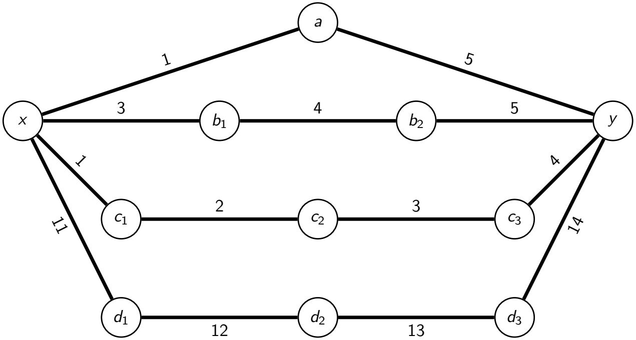

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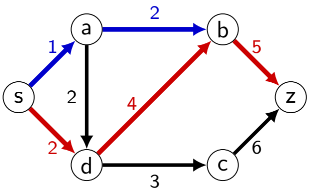
Let $P_{(s,z)}$ be an optimal temporal path and P a subpath of $P_{(s,z)}$, then, in general, P is not optimal



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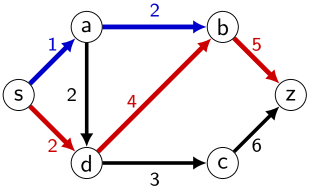

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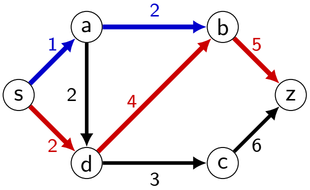

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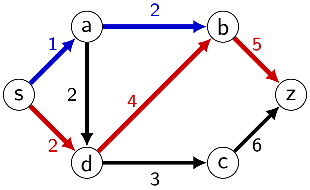

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- greedy Dijkstra does not work in general

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(Wu et al., 2014)

- source vertex x and sink v
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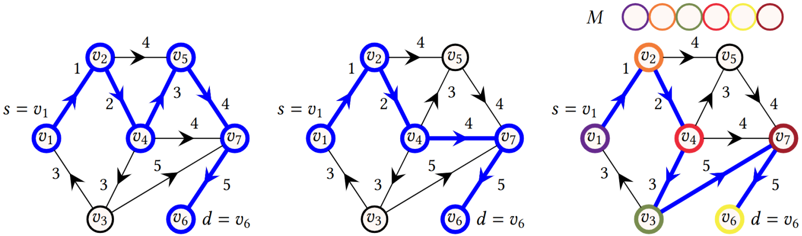
Linear time algorithms in case of equal transition times

Restless Walks and Paths

- function $\beta : V \to \mathbb{R}$ determines the maximum waiting time at each node



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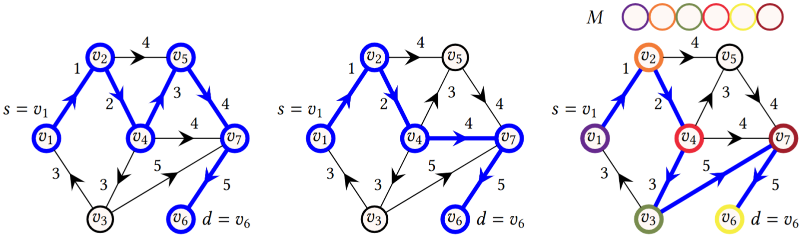

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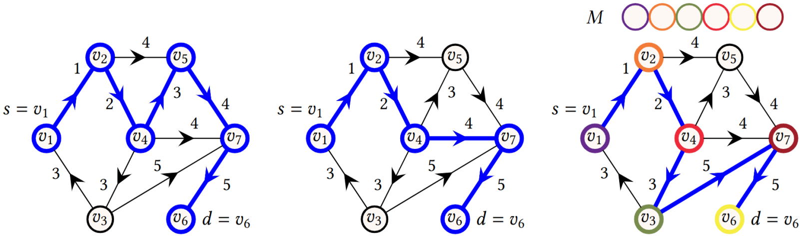

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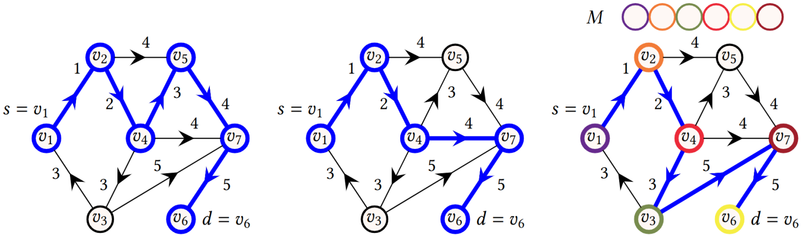

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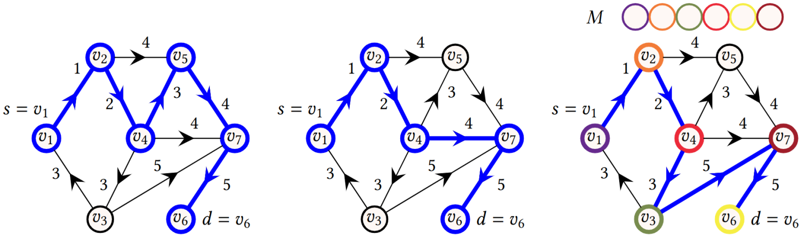
- extended for colored restless path and reachability (Thejaswi et al., 2020)



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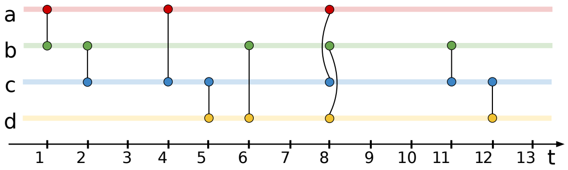

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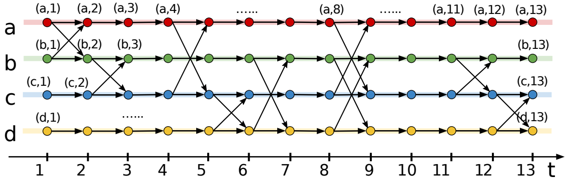

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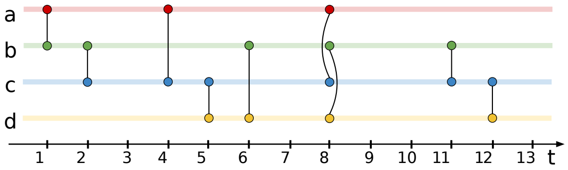

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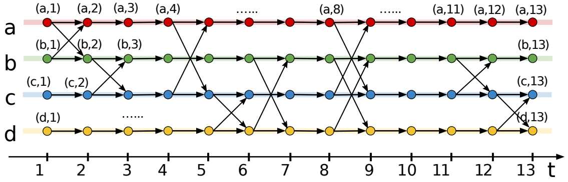

Problem: Size in number of time-steps $\Theta(|T| \cdot |V| + |E|)$

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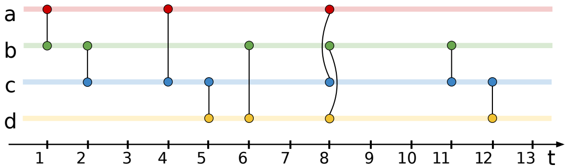
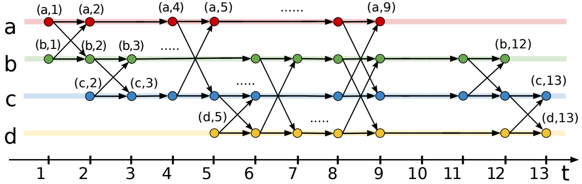


temporal graph with transition time 1



Static Expansion Graphs

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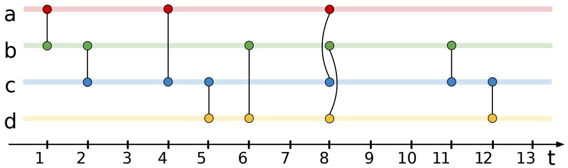

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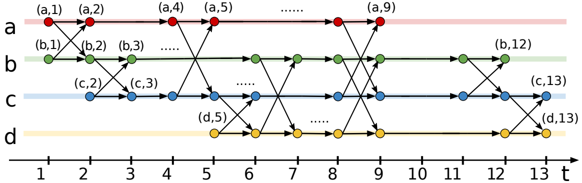
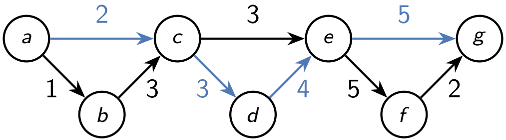


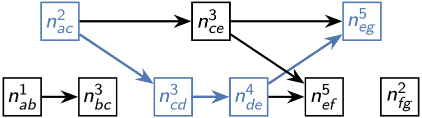
Size in number of edges $\mathcal{O}(|E|)$

Directed Line Graph

Temporal graph



Directed line graph



- temporal walk in G of length $\ell + 1 \Leftrightarrow$ walk of length ℓ in $D(G)$
- counting walks by matrix powers of adjacency matrix
- size in $\mathcal{O}(|E|^2)$

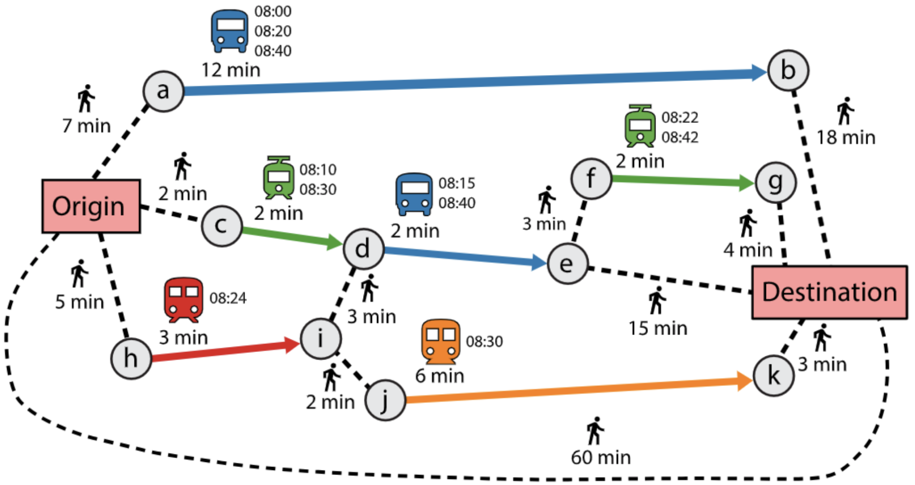
Static Representations

- static expansion graph and directed line graph are directed acyclic graphs
  if edges have non-zero transition times

- standard graph algorithms (bfs, dfs, Dijkstra, Bellman-Ford) can be adopted for finding
  - optimal temporal paths and temporal walks

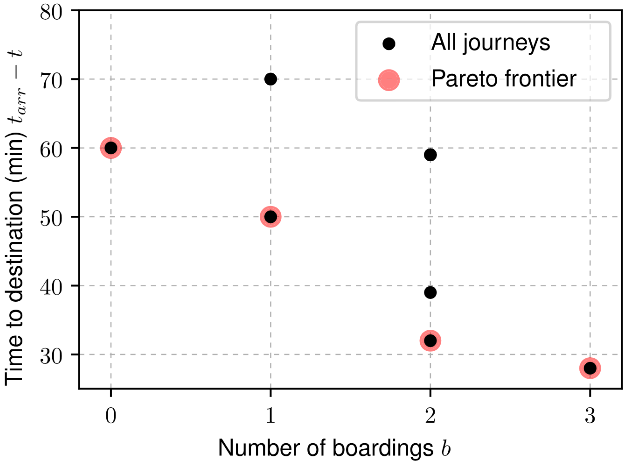
- upstream, downstream reachability sets

Transportation Temporal Networks



(Kujala et al., 2018)

Pareto-optimal Journeys



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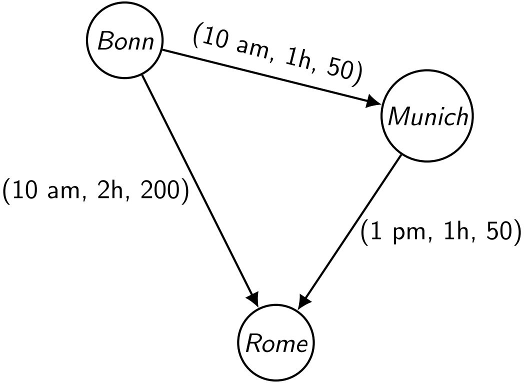
Pareto-optimal Journeys

Weighted Temporal Graph

- Additional edge costs (u, v, t, λ, c) with $c \in \mathbb{R}$



Bicriteria optimal paths

- solution: pair (duration, costs)

- non-stop: fast but expensive (2h, 200)
- via Munich: slow but cheap (4h, 100)

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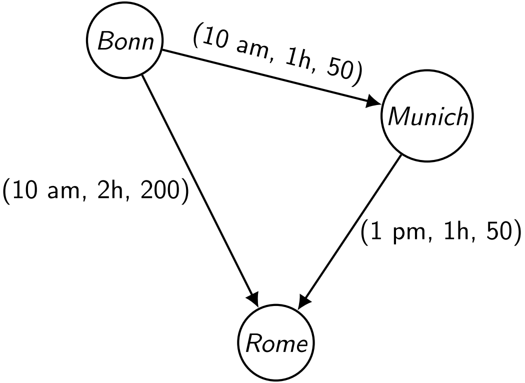

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Enumeration of temporal paths that are efficient wrt. duration and cost in polynomial delay and space

(Mutzel and Oettershagen, 2019)

Temporal Graph Properties

- many static graph properties need to be adapted for temporal graphs to be meaningful
- local and global properties, often several variants with different focus

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- diameter
  - shortest latency of time-respecting paths over connected pairs (Chaintreau et al., 2007)
  - restricted on connected pairs, as real data have many disconnected pairs

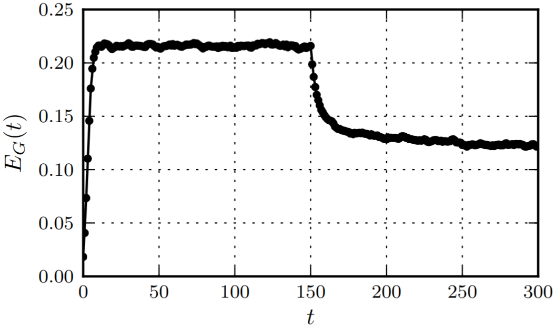
  - the minimum integer d for which the duration between each pair of nodes $u, v \in V$ is at most d (over all possible starting times) (Michail, 2016)

Temporal Network Efficiency

- network efficiency : the harmonic mean of durations (latency) over all pairs (Tang et al., 2009)

$$E(t_1, t_2) = \frac{1}{n(n-1)} \sum_{u,v \in V} \frac{1}{d_{(t_1,t_2)}(u,v)}$$

- application: robustness of network (Scellato et al., 2011)



drop in efficiency: at time $t = 150$, 20% of nodes are removed (sliding time window)

Burstiness

(Goh and Barabási, 2008)

- defined for sequence of inter-event times τ (of single node or pair of nodes, or global)
- measures deviation from memoryless random Poisson process
- defined as

$$B(\tau) = \frac{\sigma_\tau - m_\tau}{\sigma_\tau + m_\tau} \in [-1, 1],$$

where σ_τ and m_τ denote the standard deviation and mean of the inter-contact times τ

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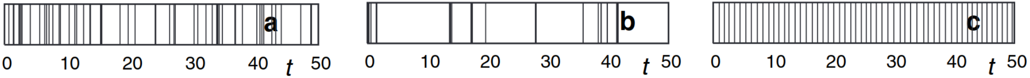
- (a) $B(\tau) = 0$ $\Rightarrow$ a Poisson distribution
  (b) $B(\tau) = 1$ $\Rightarrow$ a maximally bursty sequence
  (c) $B(\tau) = -1$ $\Rightarrow$ a periodic sequence



Topological Overlap

- quantifies the persistency of edges through time (Tang et al., 2010b)
- the topological overlap is defined as

$$T_{\mathrm{to}}(G) = \frac{1}{n} \sum_{u \in V} \frac{1}{T} \sum_{t=1}^{T-1} \frac{\sum_{v \in N(u)} \phi_{uv}^{t} \phi_{uv}^{t+1}}{\sqrt{\sum_{v \in N(u)} \phi_{uv}^{t} \sum_{v \in N(u)} \phi_{uv}^{t+1}}} \in [0, 1],$$

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- value close to zero: many edges change between consecutive time steps
- value close to one: means there are often only a few changes.

Temporal Clustering Coefficient

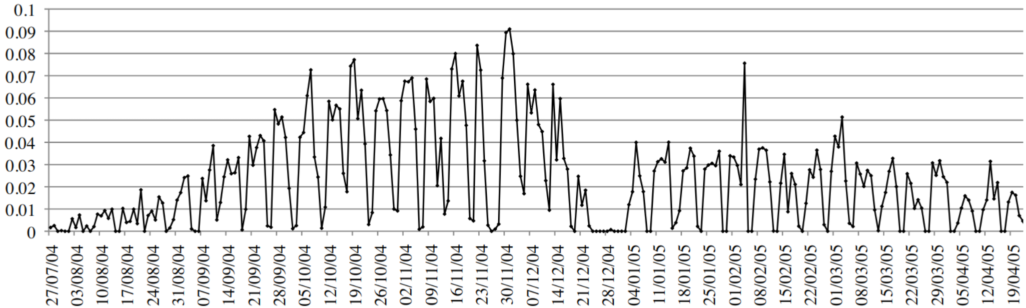
- the temporal clustering coefficient of node u in time interval I is defined as (Tang et al., 2009)

$$C_C(u, I) = \frac{\sum_{t \in I} \pi_t(u)}{|I| \binom{|N(u)|}{2}},$$

  where $\pi_t(u) =$ number of edges between neighbors of u at time t

- adaption of static clustering coefficient
- quantifies how close a nodes neighbors are to being a clique during time interval I

Temporal Clustering Coefficient



- human contact network at MIT campus using bluetooth scanning every 5 minutes
- global temporal clustering coefficient for each day
- higher during middle of the week and no clustering on holidays

(Tang et al., 2009)

Centrality Measures – Finding Important Nodes

Centrality Measures

Task

- assign to each node $v \in V$ a centrality value $c(v)$
- the higher $c(v)$ the more important is v

- many centrality measures on static graphs:
  e.g., degree, closeness, betweenness, Katz centrality, PageRank, ...

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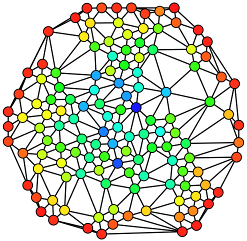

Many important applications:

- identify key players, super spreaders, important persons, ...
- ranking web pages
- H-index used for ranking academics

Temporal Centrality Measures

- many common centrality measures are walk or path based
- classification in medial and radial (Borgatti and Everett, 2006)

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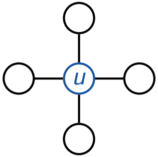
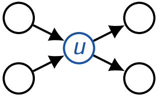
radial: captures node influence over its neighbors

- count incoming or outgoing walks or paths



medial: captures node role as intermediary

- count walks or paths passing node



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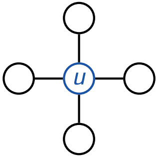

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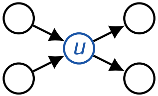

Common approach for temporal networks:
- replace path or walks with time-respecting paths or walks

Temporal Centrality

Name	Type		Use-case
temporal degree	radial	-	identify nodes with high degree
temporal closeness	radial	paths	identify nodes that can reach other nodes fast (or can be reached fast)
temporal pagerank	radial	walks	adapts static pagerank to capture concept drift
temporal katz	radial	walks	identify nodes with many incoming temporal walks
temporal H-index	radial	walks	identification of super-spreaders
temporal betweenness	medial	paths	identify nodes passed by many optimal temporal paths
temporal walk centrality	medial	walks	identify nodes that can obtain and distribute information

- choosing the right centrality measure depends on use-case
- many further temporal centrality variants, e.g., temporal eigenvector, temporal gravity, etc. (Hu et al., 2015; Rocha and Masuda, 2014; Tang et al., 2010a; Tsalouchidou et al., 2020; Bi et al., 2021; Elmezain et al., 2021; Zaoli et al., 2019; Tao et al., 2022; Taylor et al., 2021; Rozenshtein and Gionis, 2016) ...

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From Static to Temporal Closeness

Static harmonic closeness:

$$C_c(u) = \sum_{v \in V \setminus \{u\}} \frac{1}{d_s(u, v)}$$

- $d_s(u, v)$ is shortest path distance
- high centrality means short paths to many other nodes
- temporal: replace $d_s(u, v)$ with temporal distance
- several different variants
  (Wu et al., 2014; Crescenzi et al., 2020; Tang et al., 2010a; Santoro et al., 2011; Gao et al., 2015)

Temporal Closeness

Harmonic temporal closeness for $u \in V$:

$$c(u) = \sum_{v \in V \setminus \{u\}} \frac{1}{d(u, v)}$$

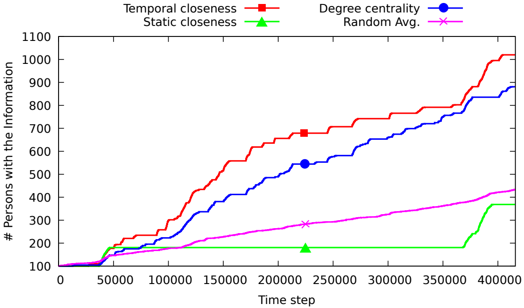
$d(u, v)$ is the minimum duration distance (i.e., arrival time - starting time).

Use case

- find nodes that spread information fast

Computation:

- call minimum duration streaming algorithm (Wu et al., 2014) for each node
- lack of scalability



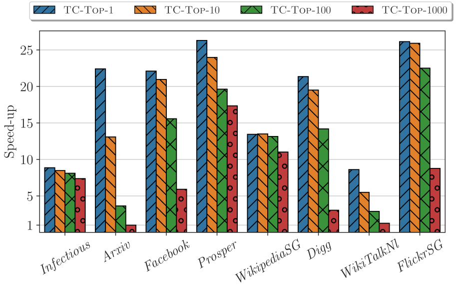
(Oettershagen and Mutzel, 2022)

Temporal Closeness – Top-k Computation

Top-k closeness problem: find all nodes with one of the k topmost closeness values

Top-k closeness computation

- for each vertex $u \in V$
  - run min. duration algorithm to compute $d(u, v)$ for all $v \in V$
    - if upper bound of $c(u)$ is smaller than k-th largest value: stop computation early



(Oettershagen and Mutzel, 2022)

Temporal Closeness – Index

Problem: rank all nodes according to temporal closeness.

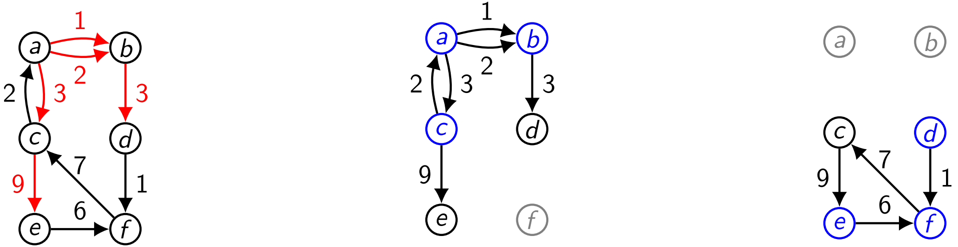
Indexing approach

- index to speed up minimum duration computation
- two phases: (i) indexing and (ii) query phase

(Oettershagen and Mutzel, 2023)

Temporal Closeness – Index

- **Construction:**
  - construct k subgraphs $\{S_1, \ldots, S_k\} = \mathcal{S}$
  - find mapping $f : V \to \mathcal{S}$ that assigns to each $S_j \in \mathcal{S}$ all vertices $v \in V$ s.t. all edges reachable from v are in $f(v) = S_j$
  - minimize size $\max_{S \in \mathcal{S}}\{|S|\}$



(a) G (b) S_1 (c) S_2

Optimal assignment is NP-hard

Temporal Closeness – Index

Approximation ratio of greedy:

$$\frac{\mathit{size}(\textsc{Greedy})}{\mathit{size}(\textsc{Opt})} \leq \frac{k}{\delta},$$

with $1 \leq \delta \leq k$ depending on the topology of the graph

Time complexity: $\mathcal{O}(nmk)$

- $n = |V|$ rounds, $m = |E|$
- each round determine S_j such that greedy choice is minimal in $\mathcal{O}(m)$ for $j \in \{1, \ldots, k\}$

Shared memory parallelization: $\mathcal{O}(\frac{nmk}{P})$ using P processors (CREW)

Temporal Closeness – Index

OOT—Out of time after 7 days. number of subgraphs $k = 2048$.

Data set	$\|V\|$	$\|E\|$	BASELINE	TOP-100	SUBSTREAM
Infectious	10 972	415 912	12.06 s	2.25 s	**1.51** s
AskUbuntu	159 316	964 437	229.73 s	132.53 s	**102.46** s
Prosper	89 269	3 394 978	1 665.20 s	260.87 s	**109.33** s
Arxiv	28 093	4 596 803	630.60 s	398.50 s	**286.86** s
Youtube	3 223 585	9 375 374	145.98 h	81.21 h	**59.72** h
StackOverflow	2 464 606	17 823 525	OOT	107.66 h	**86.49** h

- BASELINE: Streaming algorithm (Wu et al., 2014)
- TOP-100: Top-k algorithm with $k = 100$
- SUBSTREAM: Index based computation

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Static H-Index

- the H-index was originally proposed by J. E. Hirsch 2005
  $\rightarrow$ measuring the productivity and impact of scientists
- the maximum value of h such that the author has published at least h papers that have each been cited at least h times

paper ID	citations	counted in H-index
1	25	Yes
2	18	Yes
3	12	Yes
4	9	Yes
5	7	Yes
6	5	No
7	3	No

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Recently used for quantifying spreading influence (Lü et al., 2016)

Static H-Index

- $\mathcal{H}\colon \mathcal{M} \to \mathbb{N}_0$ returns for a multiset of integers $S \subseteq \{\!\{s \mid s \in \mathbb{N}_0\}\!\}$ the maximum integer i such that there are at least i elements s in S with $s \geq i$

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n-th order H-index $s_u^{(n)}$ of a node u in a static graph:

- let $s_u^{(0)} = \delta(u)$ the degree of node u, then

$$s_u^{(n)} = \mathcal{H}\left(\{\!\{s_v^{(n-1)} \mid v \in V \text{ and } v \text{ is neighbor of } u\}\!\}\right)$$

- the value of $s_u^{(1)}$ corresponds to the H-index of u

(Lü et al., 2016)

n-th Order Temporal H-Index

- the multiset $\mathcal{N}(v, t)$ contains all pairs of nodes and times (w, t_w) such that there is a temporal edge from v to w leaving at time $t' \geq t$ and arriving at time t_w

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Definition

The *n*-th order temporal H-index of a node $v \in V$ is defined as $h_v^{(n)} = h_{v,0}^{(n)}$ with

$$h_{v,t}^{(n)} = \mathcal{H}\left(\left\{\!\!\left\{ h_{w,t_w}^{(n-1)} \;\middle|\; (w,t_w) \in \mathcal{N}(v,t) \right\}\!\!\right\}\right),$$

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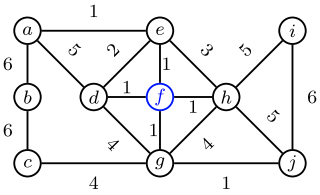
and $h_{v,t}^{(0)} = |\mathcal{N}(v,t)|$.

Computation:

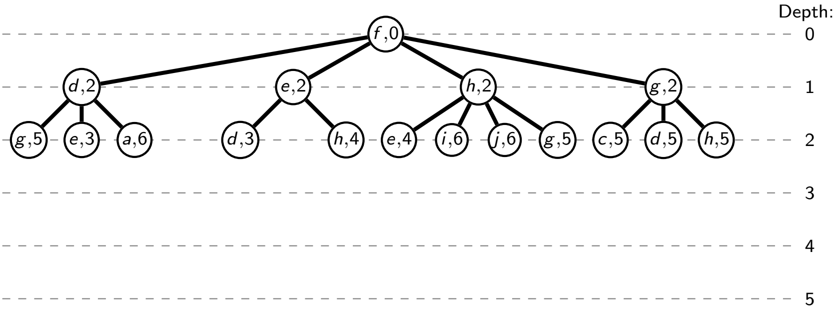
- single-pass streaming algorithm for each node i-th order H-indices for $0 \leq i \leq n$
- running time in $\mathcal{O}(|E| n \delta_{\max})$ and space in $\mathcal{O}(|V| n \delta_{\max})$

(Oettershagen et al., 2023b)

n-th Order Temporal H-Index



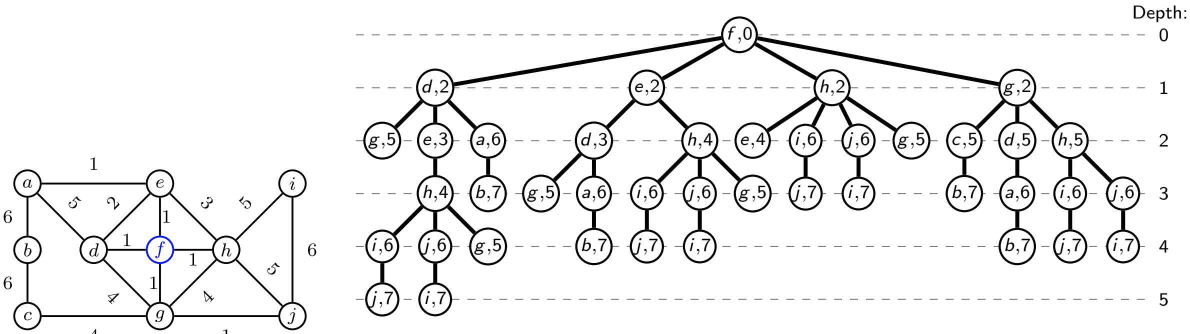
(a) Temporal network $\mathcal{G}$.



(b) The reachability tree $\Gamma(f)$ for vertex f in the temporal network shown in (a).

$$h^{(1)}_{f,0} = \mathcal{H}(\{\!\{h^{(0)}_{d,2}, h^{(0)}_{e,2}, h^{(0)}_{h,2}, h^{(0)}_{g,2}\}\!\}) = \mathcal{H}(\{\!\{3, 2, 4, 3\}\!\}) = 3$$

n-th Order Temporal H-Index



(a) Temporal network $\mathcal{G}$.

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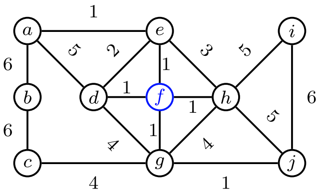
$$h^{(2)}_{f,0} =$$

n-th Order Temporal H-Index



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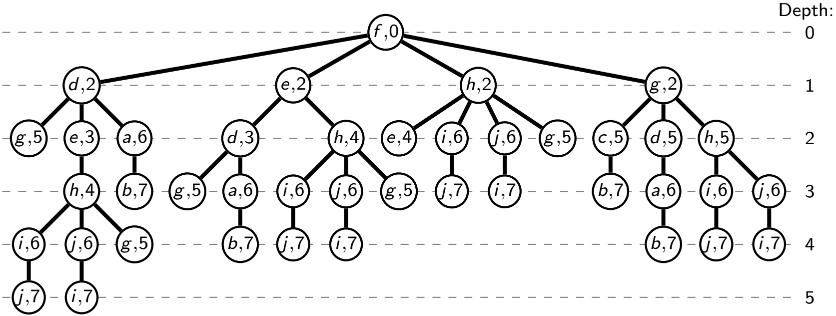

(b) The reachability tree $\Gamma(f)$ for vertex f in the temporal network shown in (a).

$$h^{(2)}_{f,0} = \mathcal{H}(\{\!\{h^{(1)}_{d,2}, h^{(1)}_{e,2}, h^{(1)}_{h,2}, h^{(1)}_{g,2}\}\!\})$$

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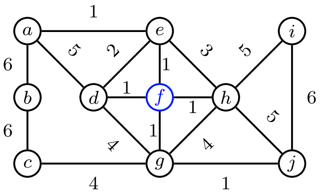

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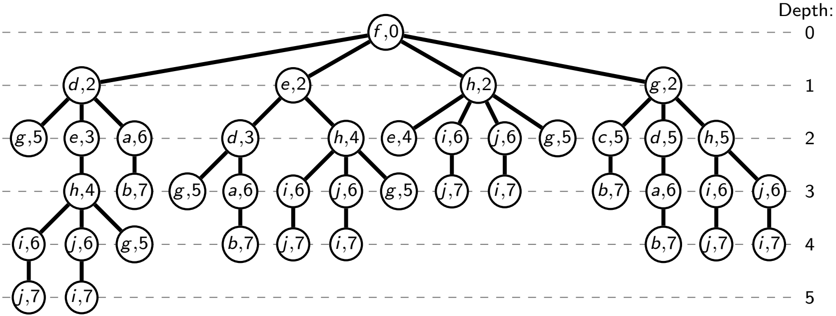

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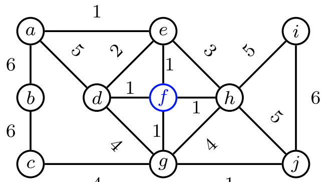

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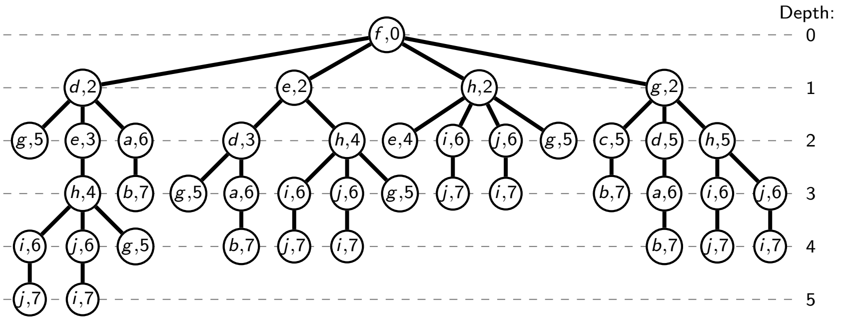

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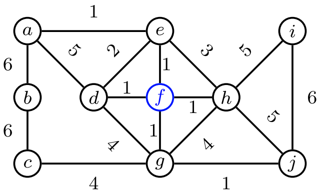

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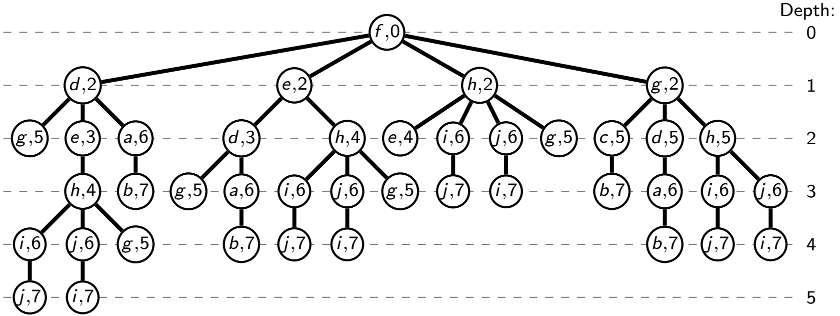

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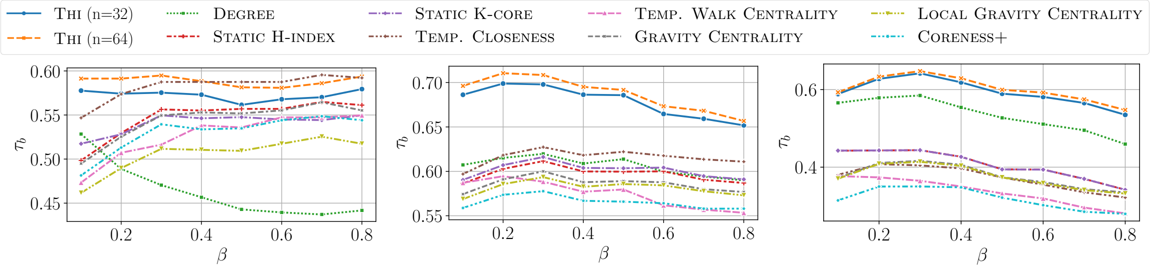
n-th Order Temporal H-Index

Use Case: Influential spreader identification

- computed for different infection probabilities β the mean node influence R_u over 1000 independent SIR simulations leading to the SIR node rankings
- compared the SIR rankings with those obtained by the centrality measures using the Kendall τ_b rank correlation measure



(a) *Malawi* (b) *FacebookMsg* (c) *Email*

Temporal Centrality

Name	Type		Use-case
temporal degree	radial	-	identify nodes with high degree
temporal closeness	radial	paths	identify nodes that can reach other nodes fast (or can be reached fast)
temporal pagerank	radial	walks	adapts static pagerank to capture concept drift
temporal katz	radial	walks	identify nodes with many incoming temporal walks
temporal H-index	radial	walks	identification of super-spreaders
temporal betweenness	medial	paths	identify nodes passed by many optimal temporal paths
temporal walk centrality	medial	walks	identify nodes that can obtain and distribute information

- choosing the right centrality measure depends on use-case
- many further temporal centrality variants, e.g., temporal eigenvector, temporal gravity, etc. (Hu et al., 2015; Rocha and Masuda, 2014; Tang et al., 2010a; Tsalouchidou et al., 2020; Bi et al., 2021; Elmezain et al., 2021; Zaoli et al., 2019; Tao et al., 2022; Taylor et al., 2021; Rozenshtein and Gionis, 2016) ...

Betweenness Centrality

- node importance of $u \in V$ in terms of number of optimal paths visiting u

$$B(u) = \sum_{s \neq u \neq z \in V} \frac{\sigma_{s,z}(u)}{\sigma_{s,z}}$$

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- computing betweenness values is at least as hard as counting optimal paths

Temporal Betweenness

Overview over the complexity of computing temporal betweenness centrality

path type	**strict**	**non-strict**
min.-hop	$\mathcal{O}(n^3 \cdot T^2)$	$\mathcal{O}(n^3 \cdot T^2)$
earliest arrival	#P-hard	#P-hard
fastest	#P-hard	#P-hard
prefix-earliest arrival	$\mathcal{O}(n \cdot m \cdot \log m)$	#P-hard
min.-hop earliest arrival	$\mathcal{O}(n^3 \cdot T^2)$	$\mathcal{O}(n^3 \cdot T^2)$

Prefix-earliest arrival: every prefix of the temporal path is an earliest arrival path

- computation using adapted Brandes' algorithm (Buß et al., 2020)

Temporal Betweenness

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  - replacing global centrality with local pass-through-degree

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Captures a nodes ability to obtain and pass on information

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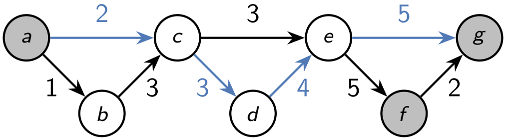
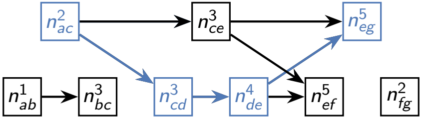


Directed line graph



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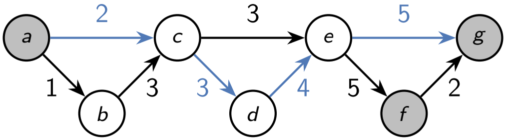

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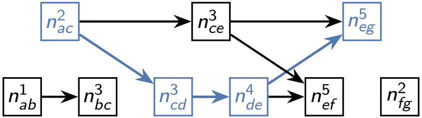

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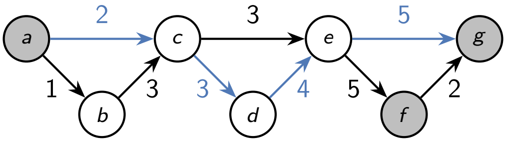

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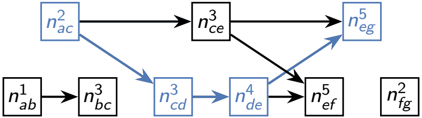

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- computation in $\mathcal{O}(|E|^{2.373})$ using matrix inversion
- approximation with power iteration in $\mathcal{O}(k(|E|^2))$

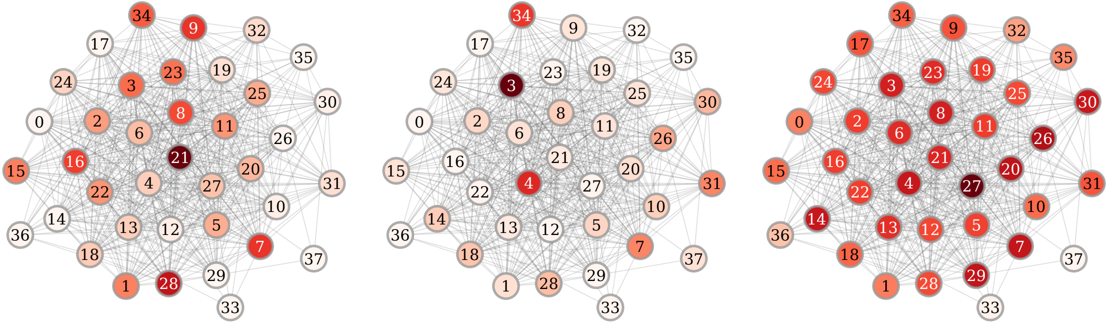
Temporal Walk Centrality

Two-Pass Streaming Algorithm

- **input:** edge sequence in chronological order, ties broken arbitrarily
- **forward pass** for computing incoming walks for $\boldsymbol{W}_{in}$
- **backward pass** for computing outgoing walks for $\boldsymbol{W}_{out}$

- running time $\mathcal{O}(|E| \cdot \tau_{max})$
- τ_{max} the largest cardinality of availability or arrival times at a node

Temporal Walk Centrality



(a) temporal walk centr. (b) temporal between. (c) static walk between.

- *enron* email subgraph: of 38 nodes and 541 temporal edges.
- colors represent centrality value: darker $\rightarrow$ higher centrality.

Temporal Centrality

- **Temporal closeness**
  - many different variants
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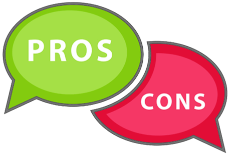

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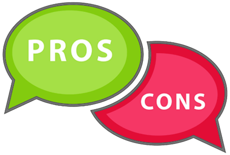

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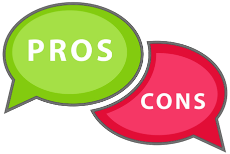

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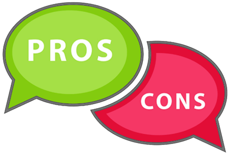

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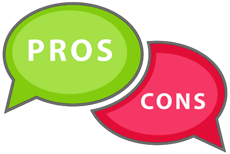

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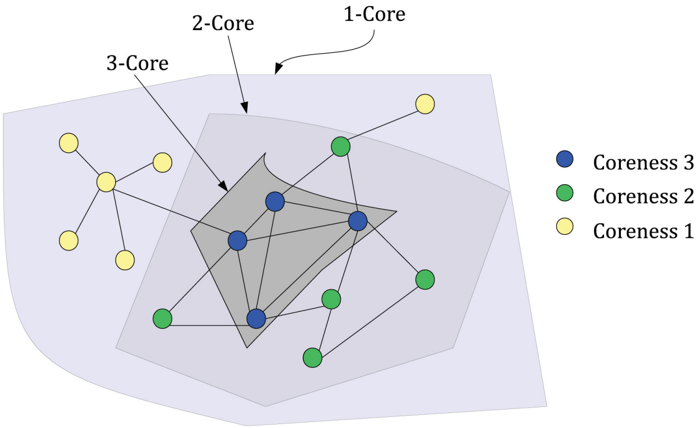
Temporal Core Decompositions

k-Core Decomposition

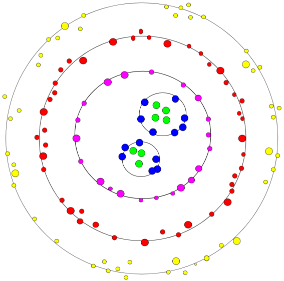
- k-core is a max. subgraph G_k of G, s.t. every node in G_k has at least k neighbors in G_k
- node u has core number $c(u) = k$ if u belongs to a k-core but not the $k+1$-core

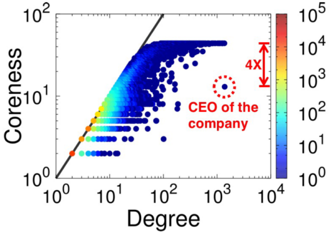


(Seidman, 1983; Kong et al., 2019)

Applications

- identifying communities and dense graphs in social networks





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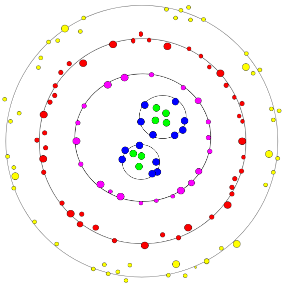

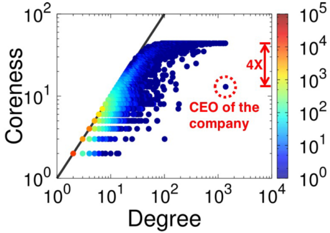

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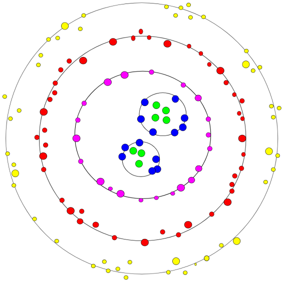

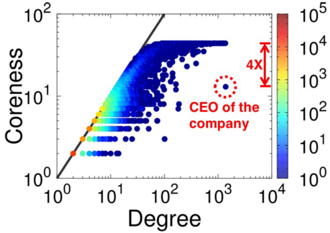

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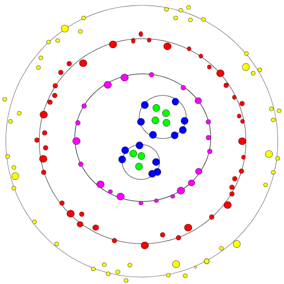

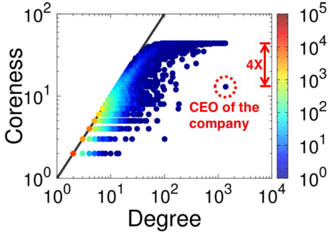

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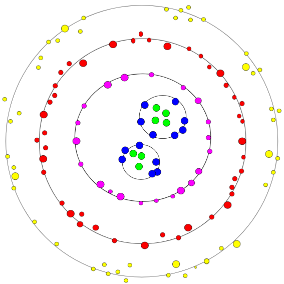

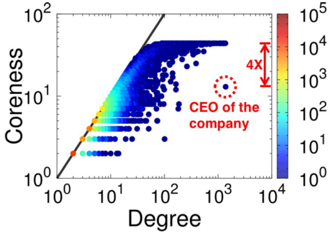

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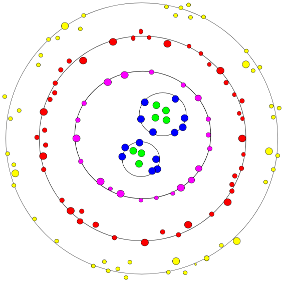
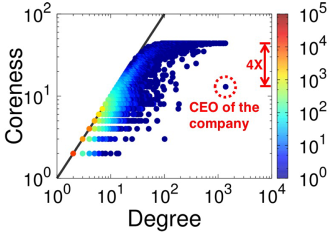
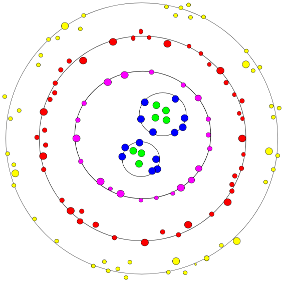


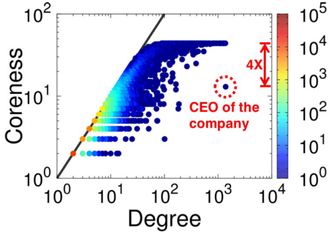


(Malliaros et al., 2020; Kong et al., 2019)

Applications

- identifying communities and dense graphs in social networks
- anomaly detection
- network visualization
- analyzing backbone structure of the internet
- network resilience and robustness
- study of spreading processes
- analyzing functional brain networks





(Malliaros et al., 2020; Kong et al., 2019)

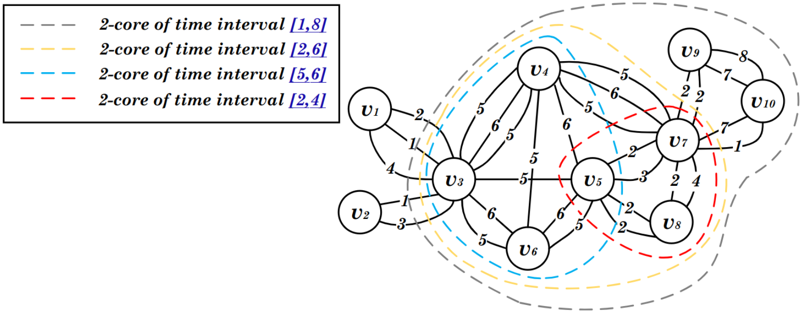
Temporal k-Core Decompositions

Variant	**Ref.**	**Running Time**	**Description**
Historical k-core	(Yu et al., 2021)	$\mathcal{O}(\log m + m_I)$	static cores spanning fixed interval I
Time-range k-core	(Yang et al., 2023)	$\mathcal{O}(\log m + \|I\| \cdot m_I)$	static cores in fixed interval
(k, h)-core	(Wu et al., 2015)	$\mathcal{O}(n + m)$	parallel temporal edges
Span-core	(Galimberti et al., 2020)	$\mathcal{O}(\|T\|^2 \cdot m)$	cores spanning intervals
(k, Δ)-core	(Oettershagen et al., 2023a)	$\mathcal{O}(m \cdot \delta_m)$	based on temporal edge degree

- more variants available (Lotito and Montresor, 2020; Hung and Tseng, 2021; Qin et al., 2022, 2020; Li et al., 2018; Oettershagen et al., 2023b) ...

- choosing the right one depends on available data and application

Historical and Time-Range k-Cores

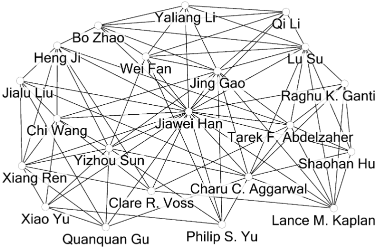


- definitions based on time window graph
  - static, aggregated graph for time window I
- historical: find a static at least k-core in time window graph for given time interval I
- time-range: find all distinct at least k-cores in all possible time windows in time interval I

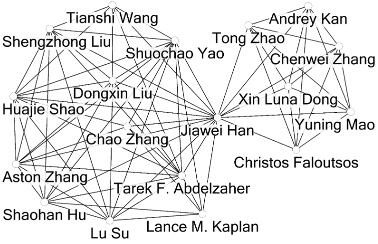
(Yu et al., 2021; Yang et al., 2023)

Historical and Time-Range k-Cores

6-cores of Prof. Jiawei Han's ego network on the DBLP snapshots



(a) *DBLP* (2011–2015)



(b) *DBLP* (2016–2020)

- straight-forward computation: restrict to interval (or subintervals)
- index-based solution to support the k-core query for every possible time window and integer k

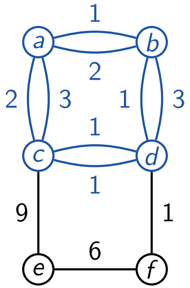
(Yu et al., 2021; Yang et al., 2023)

(k, h)-Cores

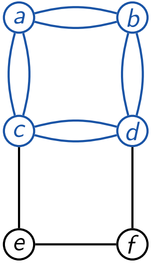
Definition

(k, h)-core is the largest subgraph H such that every v in H must have at least k neighbors in H, where each such neighbor must be connected to v with at least h temporal edges

- can be interpreted as core decomposition for multi(-layer) graphs



(a) Temporal graph G



(b) Underlying multi graph

- nodes $\{a, b, c, d\}$ induce a $(2, 2)$-core

(Wu et al., 2015)

Span-Cores

Definition

- the (k, Δ)-core is a maximal set of vertices C such that C is a k-core over the complete span of time interval Δ (each edge of the core exists in each time step in Δ)
- a span-core is maximal if no other span-core dominates it in k or Δ

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Applications:

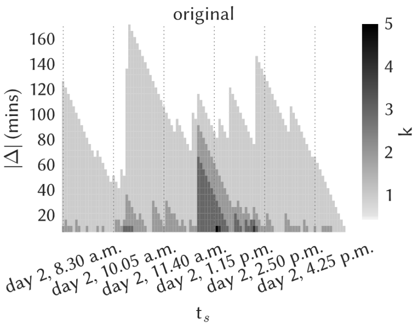
- community search
- identify temporal patterns
- anomaly detection
- graph embedding and vertex classification
- containing or maximizing spreading

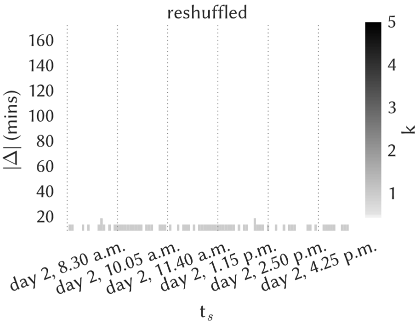
(Galimberti et al., 2020; Ciaperoni et al., 2020)

Span-Cores

Application: Temporal pattern identification

- temporal activity of a high school day





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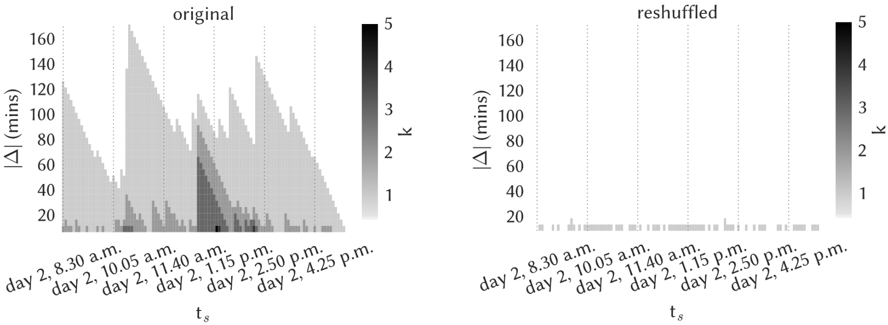

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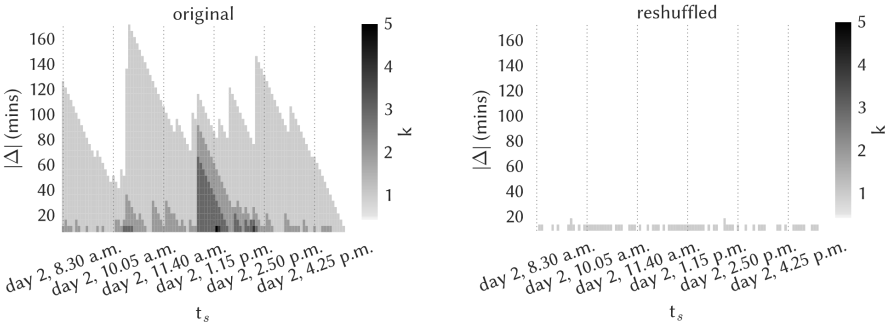
Span-Cores

Application: Temporal pattern identification

- temporal activity of a high school day
- span-core decomposition detects time-evolving cohesive substructures
- these completely disappear in the reshuffled data set



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(k, Δ)-cores

Definition

- Δ-degree of an edge is the minimum number of edges incident to one of its endpoints that have a temporal distance of at most Δ
- the (k, Δ)-core is the inclusion-maximal edge-induced subgraph $C_{(k,\Delta)}$ of G such that each temporal edge $e = (\{u, v\}, t)$ in $C_{(k,\Delta)}$ has at least a Δ-degree of $d_\Delta(e) \geq k + 1$.

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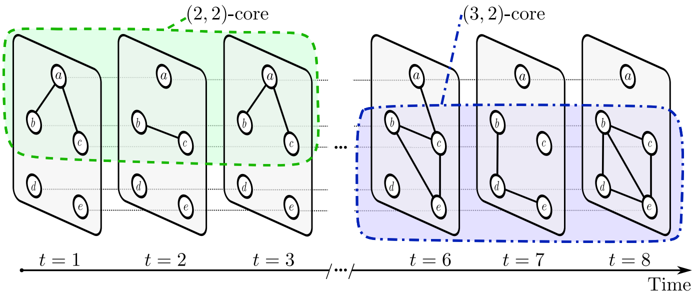


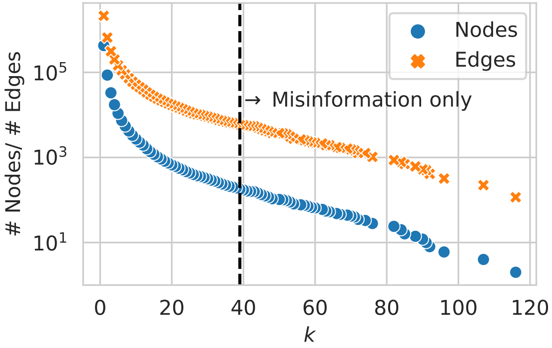
- each edge in a (k, Δ)-core is at both ends incident to at least $k + 1$ edges in the core with temporal distance at most Δ

(Oettershagen et al., 2023a)

(k, Δ)-cores

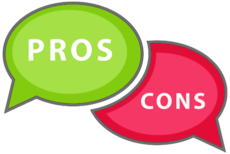
Application:

- analyzing malicious retweets in the Twitter network
- the most inner cores only contain misinformation for $\Delta = 1$ hour



Temporal k-Core Decompositions

- **Historical and time range *k*-core**
  - based on time window graphs but efficient



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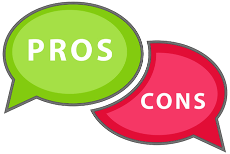

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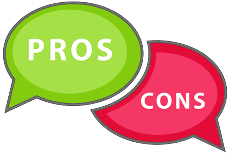
  - needs dense sequence of static graphs, also not very scalable
  - shown to be useful in wide range of applications



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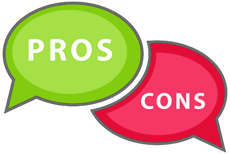

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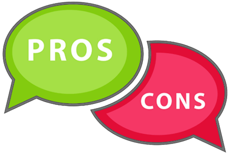

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Agenda

Part I : **Introduction and Motivation**

- models of temporal networks
- algorithmic approaches

Part II : **Mining Temporal Networks A:**

- connectivity, temporal properties
- centrality, cores

Part III : **Mining Temporal Networks B:**

- communities, patterns and events
- diffusion and random networks

Part IV : **Tools and Code Libraries**

Part V : **Challenges, Open Problems, and Trends**

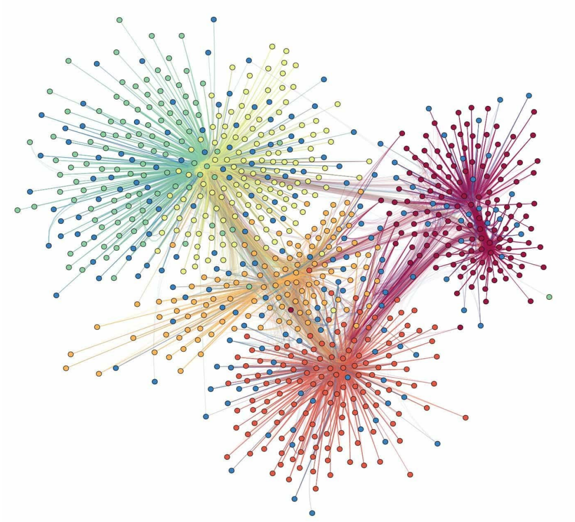
Part IV

Mining Temporal Networks B

Community Detection

Temporal Communities

Identifying communities is a fundamental task in *computer* and *network* science.



Temporal Communities

Question: How do we define a temporal community?

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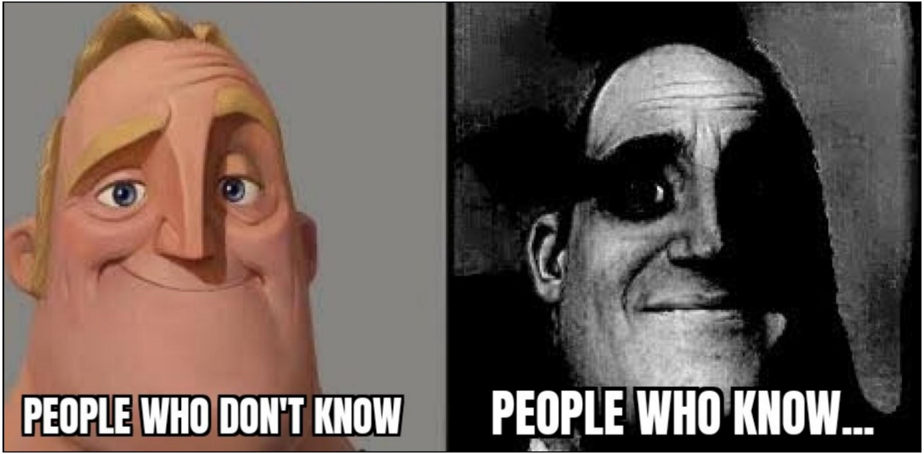

Communities in Static Networks

"community" = "umbrella term"

- extensive surveys (Fortunato and Hric, 2016; Su et al., 2024)
- possible definitions
  - a community is a set of nodes, closer to each other than to the rest of the network
  - a community is a "dense" subgraph

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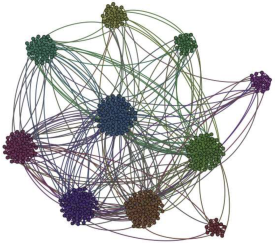
- possible definitions
  - a community is a set of nodes, closer to each other than to the rest of the network
  - a community is a "dense" subgraph
- *typical* problem settings
  - a single community vs. network partitioning
  - overlapping vs. non-overlapping communities
  - local to some nodes vs. global



Community Detection in Static Network

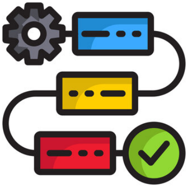
Usual workflow (data analysis)

1. pick a problem setting (e.g., partition in k communities vs identify a single local community)

2. pick a proper metric to quantify the "density" of the community S
   - average degree : $\frac{|E(S)|}{2|S|}$
   - density : $\frac{2|E(S)|}{|S|(|S|-1)}$
   - conductance : $\frac{cut(S,\bar{S})}{\min\{vol(S),vol(\bar{S})\}}$
   - modularity
   - ...

3. identify/design proper algorithms to solve the problem

4. analyze data and, if needed repeat from steps 1. or 2.



Analyses and Applications

- social networks
  - link prediction, targeted advertisement, content moderation, etc..

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- social networks
  - link prediction, targeted advertisement, content moderation, etc..
- financial networks
  - fraud detection, money-laundry activities, etc..
- collaboration networks
  - identifying trending topics, important group of nodes, etc..
- ...

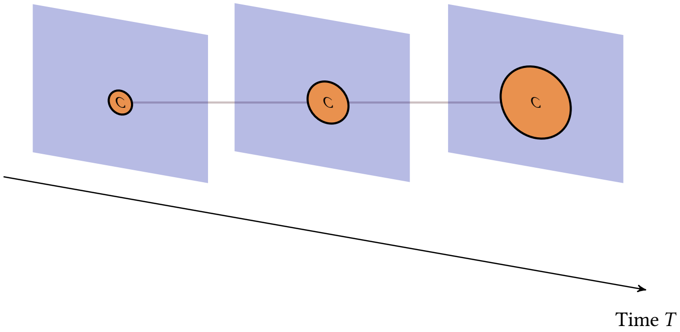
So what is new about temporal communities?

Temporal Evolution of Communities

Temporal networks allow us to study communities according to their **temporal evolution**!

Some representative behaviors,

Growing community C



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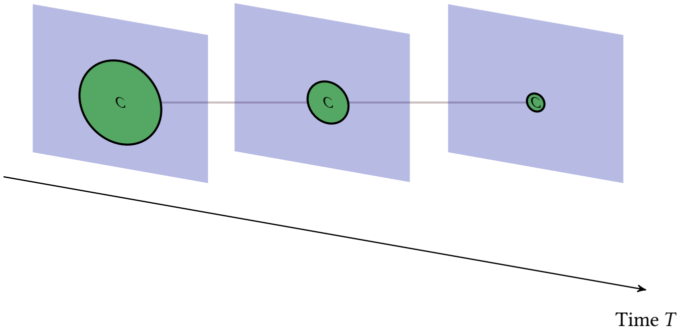

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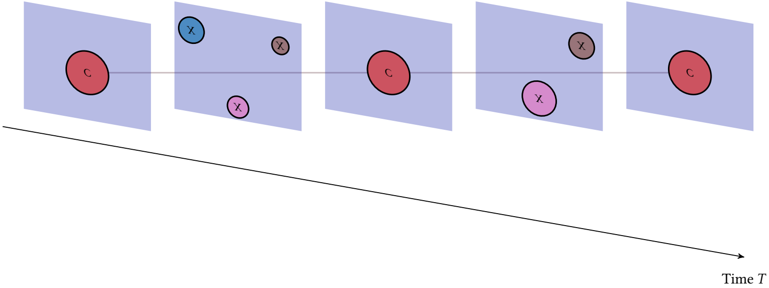

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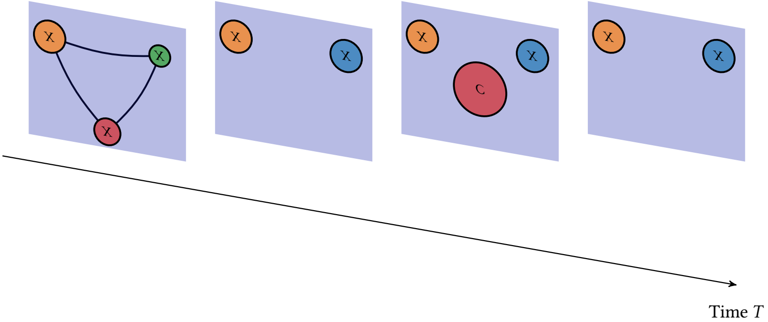

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- Shrinking community C
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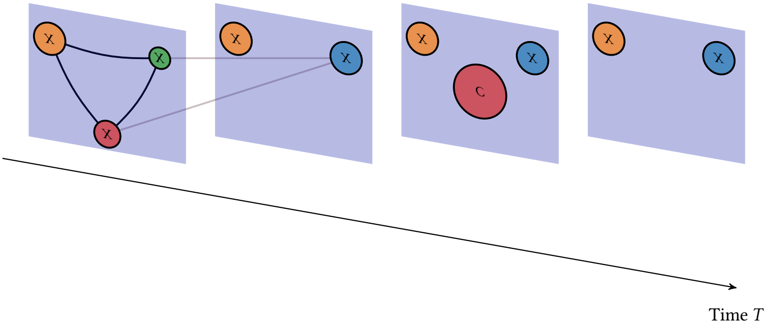
- Merging communities
- ...



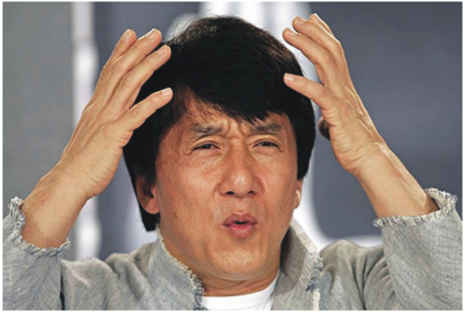
Are there proposed taxonomies?

Community Detection in Temporal Networks

Question How many taxonomies exist?

Community Detection in Temporal Networks

- proposed taxonomies
  - (Aynaud et al., 2013)
  - (Aggarwal and Subbian, 2014)
  - (Enugala et al., 2015)
  - (Renaud and Naoki, 2016)
  - (Hartmann et al., 2016)
  - (Rossetti and Cazabet, 2018)
  - (Dakiche et al., 2019)
  - (Christopoulos and Tsichlas, 2022)
  - ...



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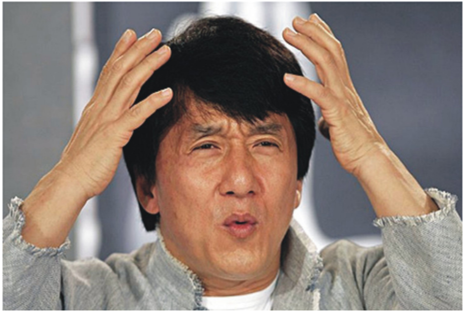
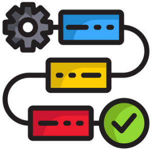


No need to panic: recall our steps for community identification!

Temporal Community Detection

Proposed workflow (exploratory analysis), everything starts from data!

1. identify if *temporal* data is fine-grained or course-grained (!)
2. pick a problem setting (✓)
3. pick a proper metric to quantify the "**temporal** *density*" of the community S, encoding the desired temporal properties (!)
4. identify/design proper algorithms to solve the problem
5. analyze data and, if needed repeat from steps 1. or 2.



Lets make it more concrete!

BFF: Finding Lasting Communities (Semertzidis et al., 2016)

Data. Consider a temporal network $G = \{G_1, \ldots, G_T\}$

Setting. We want to find a single global *dense* community across all snapshots!

Implicit assumption. time is sufficiently course-grained so each snapshot has enough information

Metrics and temporal properties. Fix a time t, given $S \subseteq V$ we define,

- $d_{avg}(S, G_t) = \frac{1}{|S|} \sum_{u \in S} d(u, G_t[S]) = \frac{2|E(S, G_t)|}{|S|}$, and
- $d_{min}(S, G_t) = \min_{u \in S} d(u, G_t[S])$

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Combining such values across snapshots, let $- \in \{avg, min\}$

- $g_{min}(d_-(S, G)) = \min_{t=1,\ldots,T} d_-(S, G_t)$
- $g_{avg}(d_-(S, G)) = \frac{1}{T} \sum_t d_-(S, G_t)$

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So we finally have a score for a community $S \subseteq V$, that is given $+, - \in \{avg, min\}$ we let

$$f_{+,-}(S, G) = g_+(d_-(S, G))$$

BFF: A Closer Look on the Metrics

Problem

Given $G = (G_1, \ldots, G_T)$, let $+, - \in \{avg, min\}$ and a target density $f_{+,-}$ find a subset of vertices $S^* \subseteq V$ **maximizing** the objective $f_{+,-}(S, G)$ over all communities.

Inspecting the objective values

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Inspecting the objective values

- $f_{min,min}(S, G) = g_{min}(d_{min}(S, G)) = \min_{t=1,\ldots,T} \min_{u \in S} d(u, G_t[S]))$,
  minimum degree on *every* snapshot of *all* vertices $v \in S^*$ is high
- $f_{min,avg}(S, G) = g_{min}(d_{avg}(S, G)) = \min_{t=1,\ldots,T} \frac{2|E(S,G_t)|}{|S|}$,
  average degree of each node in S is large on each snapshot
- $f_{avg,min}(S, G) = g_{avg}(d_{min}(S, G)) = \frac{1}{T} \sum_t \min_{u \in S} d(u, G_t[S]))$,
  on average the minimum degree of each node in S is high (more flexible than $f_{min,min}$)
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  on average the average degree of each node in S is high

BFF: Algorithm

Computing the solution, FindBFF (Inspired by Charikar (2000))

1. iteratively and greedily peel V removing at each step $v = \arg\min_{v \in V} score(v, G[V])$
2. compute the density target density on the remaining network where $V = V \setminus \{v\}$
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How is $score(v, G[V])$ computed? Depends on the objective f

- for $f_{min,min}(S, G)$ we set $score(v, G[V_i]) = \min_{t=1,\dots,T} d(v, G_t[V])$
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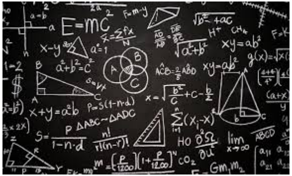
Resulting algorithms run in $\mathcal{O}(nT + \sum_t |E_t|)$

- FindBFF-MM ($f_{min,min}$), finds the optimal solution
- FindBFF-AA ($f_{avg,avg}$), finds a $\frac{1}{2}$-approximation
- BFF-AM ($f_{avg,min}$) is **NP**-hard, FindBFF achieves an approximation at most $\mathcal{O}(n^{-1})$
- BFF-MA ($f_{min,avg}$), "complexity = ?", FindBFF achieves an approximation at most $\mathcal{O}(n^{-1/2})$

On the Complexity of BFF-MA



On this line of research Charikar et al. (2018) proved

- $f_{min,avg}(S, G)$ cannot be approximated within $2^{\log^{1-\varepsilon} n}$ unless $\mathbf{NP} \subseteq \mathbf{DTIME}(n^{poly \log(n)})$
- they give $\mathcal{O}((n \log T)^{-1/2})$ and $\mathcal{O}(n^{-2/3})$ approximation algorithms
- suppose T is small the authors give and exact algorithm running in $\mathcal{O}(n^T poly(n, T))$, and a FPTAS that given an $\varepsilon > 0$ outputs a $(1 + \varepsilon)$-approximation in $\mathcal{O}(f(T) poly(n, \varepsilon^{-1}))$.

FindBFF — A Use Case

Dataset consists of publications from DBLP in years from 2006 to 2015 (each year forms a snapshot G_i), the set of nodes V ($|V| = 2625$) represents authors.

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BFF-MM	BFF-MA	BFF-AM	BFF-AA
Wei Fan, Philip S. Yu, Jiawei Han, Charu C. Aggarwal Lu Qin, Jeffrey Xu Yu, Xuemin Lin Craig Macdonald, Iadh Ounis	*Wei Fan*, Jing Gao, *Philip S. Yu, Jiawei Han, Charu C. Aggarwal* Jeffrey Xu Yu, Xuemin Lin, Ying Zhang	*Wei Fan*, Jing Gao, *Philip S. Yu, Jiawei Han*	*Wei Fan, Jing Gao, Philip S. Yu, Jiawei Han, Charu C. Aggarwal,* Mohammad M. Masud, Latifur Khan, Bhavani M. Thuraisingham

Some observations

- Not all authors appearing in a dense subset coauthored many papers together, e.g., "Wei Fan", "Philip S. Yu", and "Jiawei Han" coauthored only two papers together but many pairwise.
- Some solutions are *not connected*, e.g., BFF-MM.

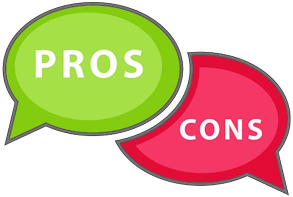
Pros and Cons

Strengths

- computes dense communities based on degree scores
- it is efficient for certain formulations
- works well for sufficiently temporal course-grained data

What is missing

- not optimizing for specific time-frames of the time-span
- cannot adapt well for fine-grained data
- theoretical gaps may prevent (close to) optimal solutions



Lets see *another formulation*!

Communities as Episodes (Rozenshtein et al., 2019)

Data. Let $G = (V, E)$ be an undirected temporal graph with $E = \{(u_i, v_i, t_i) : i = 1, \ldots, m\}$.

Setting. Find a multiple *dense* communities covering the timespan $[t_1, t_m]$

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Metrics and temporal properties.
Given an interval $I = [t_s, t_e] \subseteq [t_1, t_m]$ let $G[I] = (V[I], E[I])$ be the induced subgraph in I

An *episode* is a pair $(I = [t_s, t_e], H \subseteq G[I])$ where H is a subgraph of $G[I]$

Given a static subgraph $H = (V_H, E_H)$ its density is $d(H) = \frac{|E_H|}{|V_H|}$

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Problem

Given a temporal network G and an integer k find a set of k episodes $(I_i, H_i), i = 1, \ldots, k$ such that $I_i \cap I_j = \emptyset, i \neq j$ are disjoint while maximizing

$$\sum_{i=1}^{k} d(H_i)$$

Solving the Problem trough Segmentation

Computing the solution

- **Note!** in an optimal solution $\cup_i I_i$ covers $[t_1, t_m]$.
- given interval I use result in (Goldberg, 1984) in $O(nm_{st} \log n)$ to obtain $H^* = \arg\max_{H \subseteq G[I]} d(H)$, and m_{st} is the max edges in an interval
- it suffices to find an optimal segmentation $I_1, \ldots, I_k$, (with dyn. prog. in $\mathcal{O}(kT^2)$)

Such algorithm has runtime $\mathcal{O}(kT^2 nm_{st} \log n)$

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Such algorithm has runtime $\mathcal{O}(kT^2 nm_{st} \log n)$

As this can be prohibitive an approximate approach is proposed

- *approximate dyn. prog.* approach controlled ε_1 pruning candidates in segmentation
- avoiding recomputing densest subgraph, use *evolving solution*, controlled by ε_2

Leading to $2(1+\varepsilon_1)(1+\varepsilon_2)$-approximation algorithm running in $\mathcal{O}(\frac{k^2}{\varepsilon_1\varepsilon_2} T m_{T,max} \log^2(n))$ where $m_{T,max}$ is the maximum number of edges with the same time-stamp.

Communities as Episodes — Analyzing Twitter Data

Tweets in Helsinki region, V are hashtags and (u, v, t) is added if a message contains both hashtags u, v, data gathered from Nov 2013 containing 4758 tweets and 917 nodes. (Parameters, $k = 4$ and $\varepsilon_1 = \varepsilon_2 = 0.1$)

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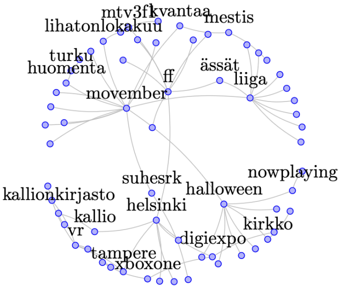
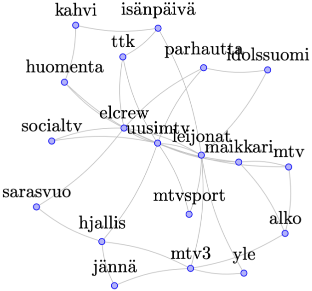


01.11 03:31 – 02.11 11:00



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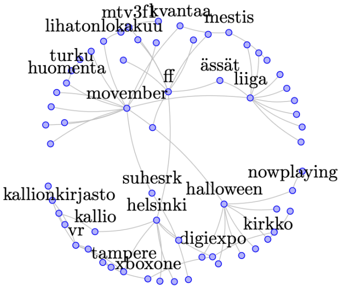

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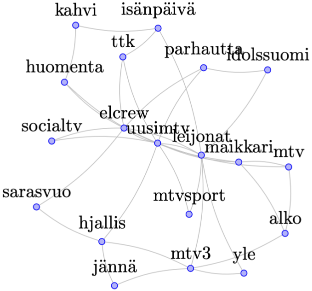

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- both episodes are related to events occurring in that period in Helsinki and globally, e.g., on the left the Digiexpo and Halloween events, while on the right the MTV Europe music awards that was on November 10.

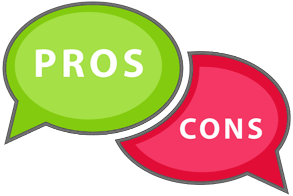
Pros and Cons

Strengths

- it adapts in a flexible way to the timespan of the network
- in general it is efficient to compute
- identifies dense communities (w.r.t., average degree)

What is missing

- when aggregation is performed may lose some information
- not accounting for temporal behaviors, e.g., periodicity or burstiness
- may not be informative with small timespan (i.e, G has not few dense structures)



Lets see another formulation!


What about
Social Networks

Discovering Buzzing Stories (Bonchi et al., 2019)

Data. Given a temporal network $G = (V, \{E_t, f_t\}_{t=1,\ldots,T})$.

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- V is a set of objects, e.g., *messages or words* from chat apps or posts on socials.

- E_t edges at time t and $f_t : E_t \mapsto \mathbb{R}^+$ is a weighting function capturing the strength between two objects at t, e.g., $f_t(\text{cream}, \text{carbonara})$

Discovering Buzzing Stories (Bonchi et al., 2019)

Data. Given a temporal network $G = (V, \{E_t, f_t\}_{t=1,\dots,T})$.

Setting. Find multiple *dense* and unexpected *stories*, i.e., communities.

Metrics and temporal properties.

Transform $G \to G_A = (V, \{E_t, \phi_t\}_{t=1,\dots,T})$ by mapping $f_t \to \phi_t$ where $\phi_t(\cdot)$ captures how anomalous or unexpected is e based on past data.

Given a *discrete*-time interval $I = [t_s, t_e]$ and $S \subseteq G_A$ then "$\delta(S, I)$ = density of S in I"

Given a set of subgraphs $\mathcal{S}$ then $\Delta(\mathcal{S}, I) = \sum_{S \in \mathcal{S}} \delta(S, I)$.

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Problem

Given the temporal (anomalous) graph G_A, an interval $I \subseteq [t_1, t_m]$ and two integers $K, N \geq 1$ find $\mathcal{S}^* = \{S_1, \dots, S_K\}$ of disjoint subgraphs of G_A such that

- $|S_i| \leq N$, that is the number of nodes in each subgraph is bounded by N
- $\Delta(\mathcal{S}^*, I)$ is maximized.

Discovering Buzzing Stories — Algorithm

The decision problem is **NP**-hard

Computing a solution

- develop an *exact* algorithm $\mathcal{A}$ running in $\mathcal{O}(|I| m \log n)$ for $K = 1$ and $N = \infty$ based on a "peeling" the minimum degree vertices.

  Note! This needs to be accounting for time-points (updating the vertex degrees).

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- algorithm for general case using $\mathcal{A}$ as subroutine by
  - imposing further *constraints on the size* (i.e., bounding N) of the reported solutions
  - iteratively *remove identified communities* to guarantee disjoint output.

The resulting algorithms runs in $\mathcal{O}(K|I|m \log n)$ and comes without guarantees.

Discovering Buzzing Stories – A Use Case

Dataset. Yahoo searches during 2013-2014

Assumption. If there is an anomaly people will search it on the web!

Processing.

- dataset spans 558 days, is build on user queries (appearing at least 50 times per day)
- map queries e.g., "How to put pineapple on pizza" $\rightarrow$ "(pineapple,pizza)" is generated
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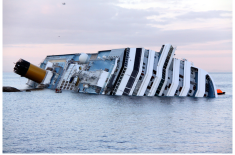
Date	$\|I\|$	N	**Story**	**Related Event**
13/01/2014	1	10	"cristiano dor wins ronaldo fifa ballon"	Cristiano Ronaldo won the Ballon d'Or in 2014
09/02/2014	3	10	"day figure russia julia skating medal ceremony"	Yulia Lipnitskaya, a Russian prodigy won gold medal in Sochi
27/02/2014	2	30	"captains costa wreck concordia"	Legal process for the Costa Concordia disaster



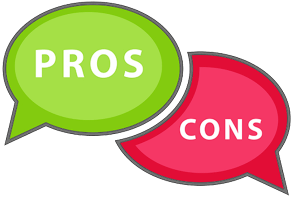
Pros and Cons

Strengths

- tailored to a specific application
- interesting analyses
- efficient algorithm in practice

What is missing

- no guarantees
- a lot of preprocessing is needed and identifying $\phi(\cdot)$ may be non-trivial
- not much used in practice




TIME FOR...
QUESTIONS

Significant Engagement Based Community (Zhang et al., 2022)

Dataset. Undirected temporal network $G = (V, E), E = \{(u_i, v_i, t_i) : i = 1, \ldots, m\}$

Setting. Find the community where a given **user** has highest engagement, *local formulation*!

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Metrics and temporal properties.
Let $H \subseteq G$, the *degree* of $v \in V[H]$ is $d_{u,H} = \sum_{e:E[H]} \mathbf{1}[v \in e]$.

Given $H \subseteq G$ the *engagement* of $v \in V[H]$ is $\gamma(u, H) = \frac{d_{u,H}}{\sum_{v \in V[H]} d_{v,H}}$.

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Problem

Given a temporal graph G, a parameter $k \geq 1$, and a vertex $u \in V$ find H such that

- $u \in V[H]$
- the static network of H is a k-core (Guarantees cohesiveness)
- $\gamma(u, H) \geq \gamma(u, H')$ for any other $H' \subseteq G$ (Guarantees max-engagement)

A Case Study on DBLP (Zhang et al., 2022)

Computing a solution

Greedy *peeling* algorithms + local search running in $\mathcal{O}(m^2(n+m))$

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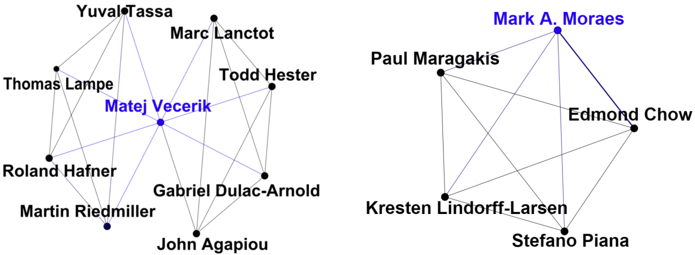
Insights on DBLP data



As desired seed nodes are well centered in the identified communities

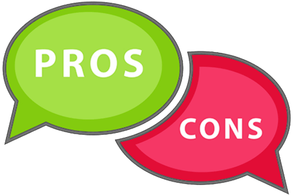
Pros and Cons

Strengths

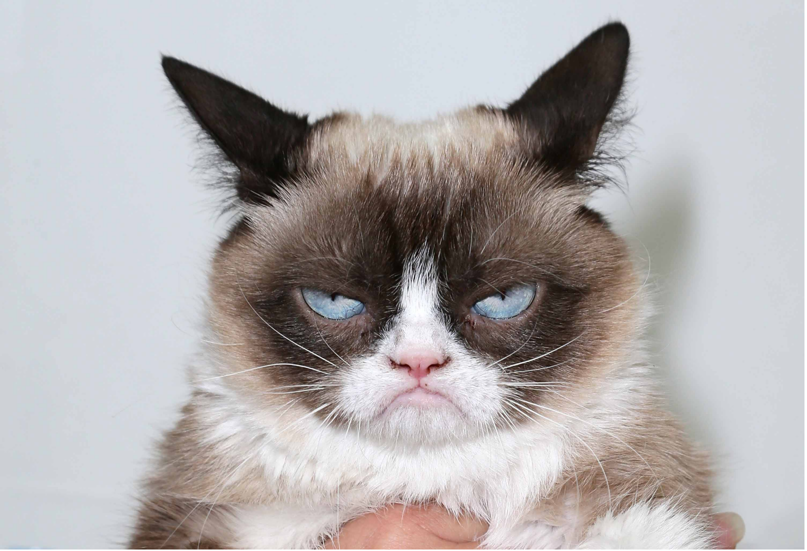
- local formulation
- proposed algorithm is polynomial
- output has desired properties

What is missing

- no guarantees
- may need additional assumptions to better model engagement
- engagement is not time-dependent



You showed us only degree-based metrics!



Local Motif Clustering (Fu et al., 2020)

Data. Undirected temporal network $G = \{G_1, \ldots, G_T\}$.

Setting. Find a good local "*tight cluster* $\to$ *motifs*" to a *seed node* at each time point $1, \ldots, T$.

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Metrics and temporal properties.

H: a small subgraph pattern (e.g., edge, triangle, star etc).

$C \subseteq V$: is cluster, and *motif-conductance* is $\Phi(C, H) = \frac{\partial(C,H)}{\min\{vol(C,H), vol(\bar{C},H)\}}$ where,

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- $vol(A, H)$: $\#$ of occurrences of H in A

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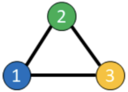
Problem

Given $G = \{G_1, \dots, G_T\}$, a static motif H, a seed node $v \in V$, an upper-bound on the motif conductance ϕ, compute,

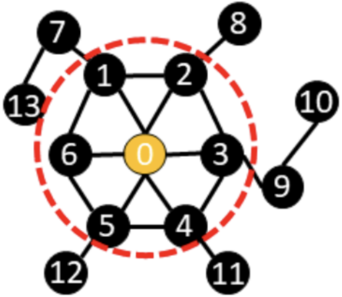
$$C^t \text{ containing } v \in V \text{ such that } \Phi(C^t, H) \leq \phi, \text{ for each } t = 1, \dots, T.$$

Local Motif Clustering – Example

Example

Let us fix H to be a triangle 

Notation. Blue edges denote insertions and green edges denote removals.



(a) $t = 1$

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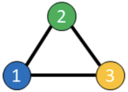
Let us fix H to be a triangle 

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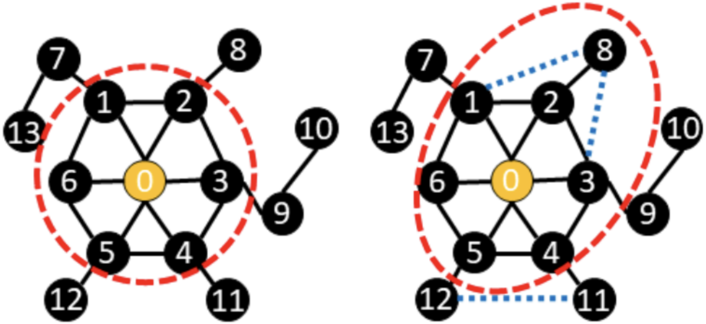

(a) $t = 1$ (b) $t = 2$

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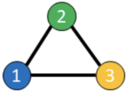
Let us fix H to be a triangle 

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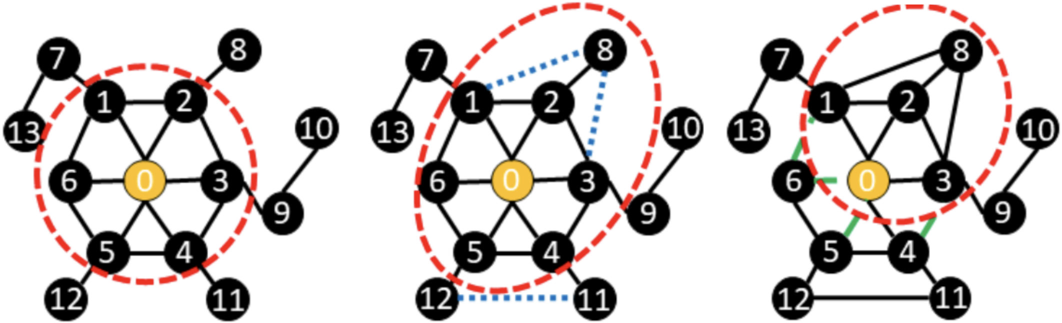

(a) $t = 1$ (b) $t = 2$ (c) $t = 3$

Proposed Solution by Fu et al. (2020)

The problem is already for static graphs **NP**-hard 🙁

Proposed Solution by Fu et al. (2020)

The problem is already for static graphs **NP**-hard 🙁

We will be looking for an approximate solution

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Computing a solution

- fix $t = 1, \ldots, T$, let k be $\#$ nodes in H
- define a multilinear page-rank vector $\mathbf{x}^t$, that accounts for the high-order motifs,

$$\mathbf{x}^t = \alpha P^t(\underbrace{\mathbf{x}^t \otimes \cdots \otimes \mathbf{x}^t}_{k-1 \text{ times}}) + (1-\alpha)\mathbf{u}$$

where P^t encodes transitions over motifs, $\mathbf{u}$ is the vector encoding user preferences.

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- a good-approximate obtained through sweep cut on vector $\mathbf{x}^t$,
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Idea! For varying t, *graph is evolving* avoid repeating such steps by **1.** avoid considering distant edges that cannot impact cluster **2.** avoid scratch re computation of $\mathbf{x}^t$

Runtime $\mathcal{O}(\sum_t [f(m_i, n_i^{O(k)}, k^k)]$, P^1 is assumed in input!

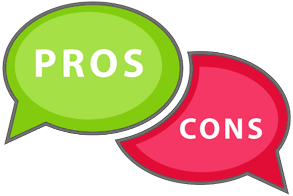
Pros and Cons

Strengths

- local formulation
- uses high order information
- versatile according to the pattern

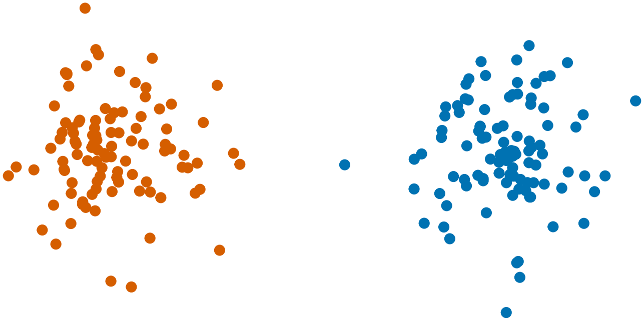
What is missing

- no guarantees
- problem is already hard on a single snapshot
- not very practical computing $P^i, i = 1, \ldots$



Fairness-Aware Clique Preserving Clustering (Fu et al., 2023)

Fairness definition demographic fairness



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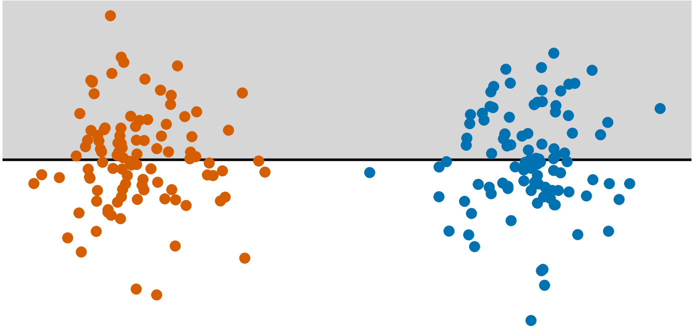

Fairness-Aware Clique Preserving Clustering (Fu et al., 2023)

Data. $G = \{G_1, \dots, G_T\}$, where V has h different groups

Setting. Find q clusters at each time-point cutting few k-cliques and that are fair.

Problem

Given G parameters k and q, find a q-clustering $C_1^t, \dots, C_q^t$ for $t = 1, \dots, T$ such that

- $\min_{C_i^t} \sum_{t=1}^{T} \sum_{j=1}^{q} \frac{\partial(C_i^t, k)}{vol(C_i^t, k)}$ ($\partial(C_i^t, k)$: $\#$ of k-cliques cut by C_i^t)
- $\frac{|V_S \cap C_i^t|}{|C_i^t|} = \frac{|V_s|}{|V|}$ for each time $t = 1, \dots, T$ and cluster $s = 1, \dots, q$

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- $\frac{|V_S \cap C_i^t|}{|C_i^t|} = \frac{|V_s|}{|V|}$ for each time $t = 1, \dots, T$ and cluster $s = 1, \dots, q$

Computing a solution

The problem is **NP**-hard

At fixed t solution is based on spectral techniques: trace minimization problem $+$ K-means.

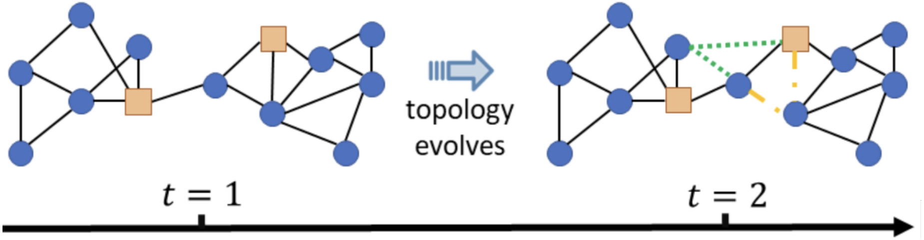
Avoid re computation at each time t, accounting for edge additions and deletions.

Runtime is $\mathcal{O}(T(q^4 + q^2 n) + \sum_t k a^{k-2} m^t)$, a is arboricity at time t, no guarantees on the solution.

Visualizing the Desired Behavior

Example

Let $q = 2$ and $k = 3$ (k-clique is a triangle). **Green edges** are insertions and **yellow edges** are removals



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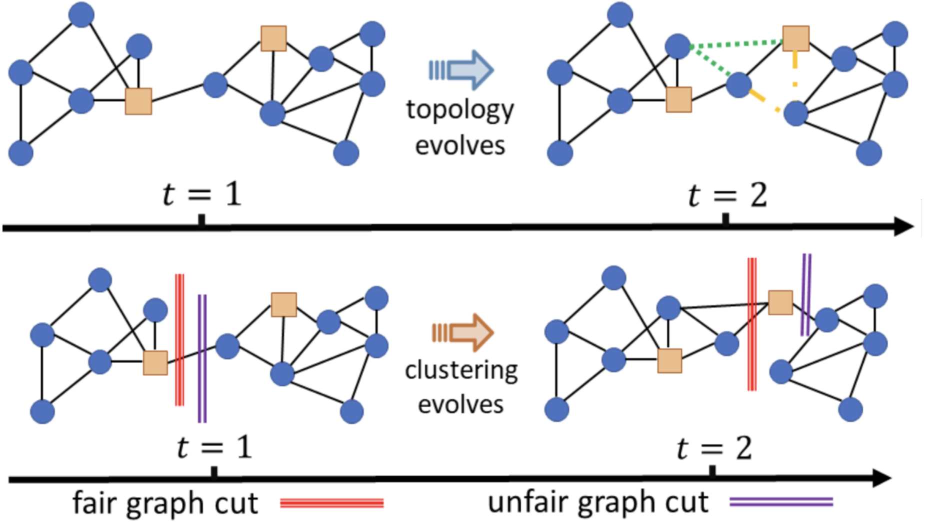


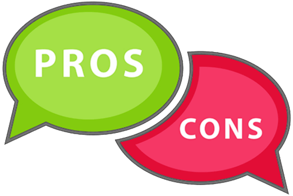
Pros and Cons

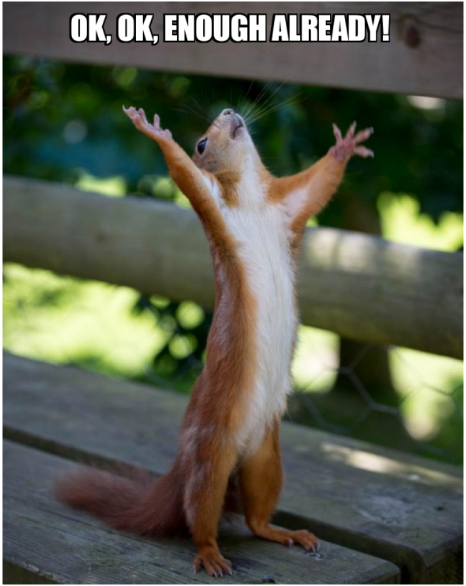
Strengths

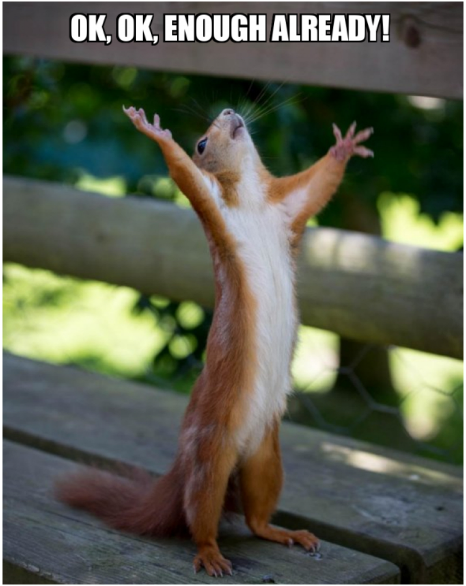
- fairness + evolving networks
- global dense clusters
- can have different applications

What is missing

- no guarantees
- problem is already hard on a single snapshot
- not very practical enumerating k-cliques




OK, OK, ENOUGH ALREADY!



Let us do a summary.

Community Detection in Temporal Networks – Summary

- **community detection in temporal networks** is a very wide research area
- do not panic and follow a **principled approach** (start from data!)
  - identify **properties of the communities** you are looking (global vs. local, etc..)
  - **search formulations** with desired properties (*much work has been done!*)
  - if nothing works you found a **gap in literature** (novel algorithms are needed!)

- **use/develop proper algorithms** to analyze temporal communities

Keep in mind. There is a **gap between formulations and applications**

Temporal communities – Other Formulations

Some other existing formulations

- (Lin et al., 2022) find multiple maximal quasi-clique based communities, stable overall and with interval-based edges
- (Qin et al., 2023) find single and dense community that is periodic over time
- (Preti et al., 2021) discovering a set of diverse and correlated communities in dynamic setting
- (Ma et al., 2020) finding dense subgraphs in temporal networks with time-varying edge weight
- (Banerjee and Pal, 2022) online algorithm for temporal clique identification
- (Chu et al., 2019) bursty and dense community identification
- ...


Coffee time
Right meow
quickmeme.com

Temporal Motifs and Events

Subgraph Motifs

Motifs are small **subgraph patterns** with a plethora of applications in various domains

- databases
- biology
- social networks
- e-commerce

Subgraph Motifs

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- databases
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But how?

Short Primer on Subgraph Isomorphism

Given a simple static graph $G = (V, E)$ and small target graph $H = (V_H, E_H)$ we say that,

- $G' \subseteq G$ is a subgraph that is isomorphic to H if there exists a bijection $f : V_H \mapsto V_{G'}$ with

$$(x, y) \in E_H \Longrightarrow (f(x), f(y)) \in E_{G'} \qquad (G' \sim H)$$

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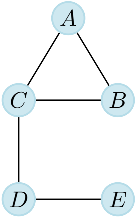
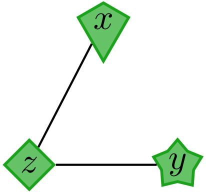
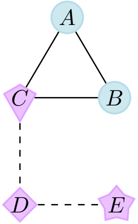


(a) G



(b) H

Is "$V[C, D, E]$" isomorphic to H?



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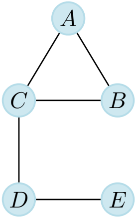

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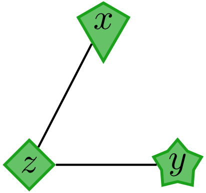

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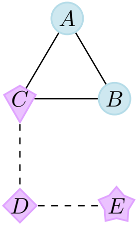

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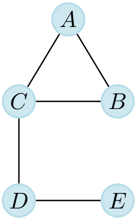

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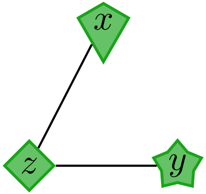
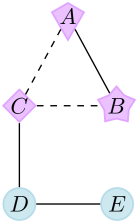


(b) H

Is "$V[A, C, B]$" isomorphic to H?



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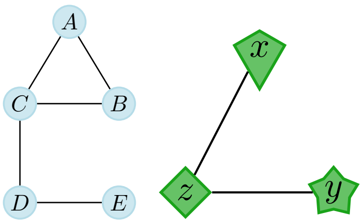

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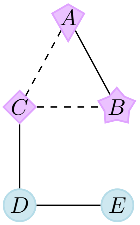

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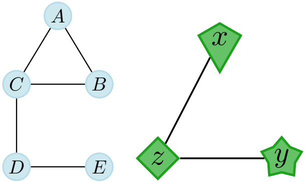
Example



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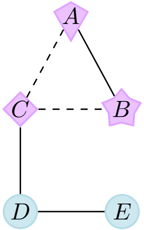

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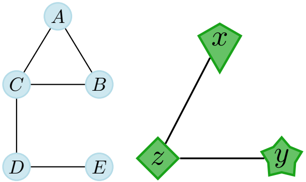
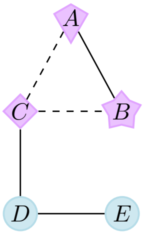


(a) G (b) H

Is "$V[A, C, B]$" isomorphic to H? **No!**



$(f^{-1}(A), f^{-1}(B))$ not in H!

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Given a graph G, count of H meaning: $\#$ of distinct subgraphs $G' \subseteq G$ with $G' \sim H$.
If $G' \sim H$ we say G' is an *occurrence* of H

Problem

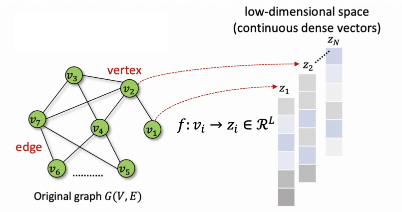
Given a graph G and a *small* subgraph pattern H

- obtain the count of H (counting problem)
- list all occurrences of H

Subgraph Counts

The problem is **NP**-hard and extremely challenging

Many applications, in *computer science*, *network science* and more...



(a) Node embeddings

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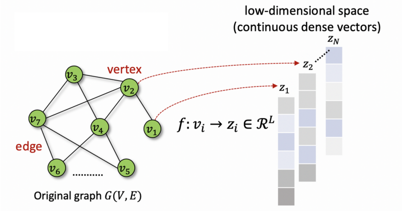

(a) Node embeddings

(b) Malware detection

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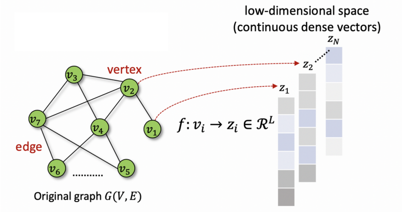
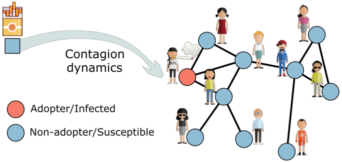


(a) Node embeddings

(b) Malware detection



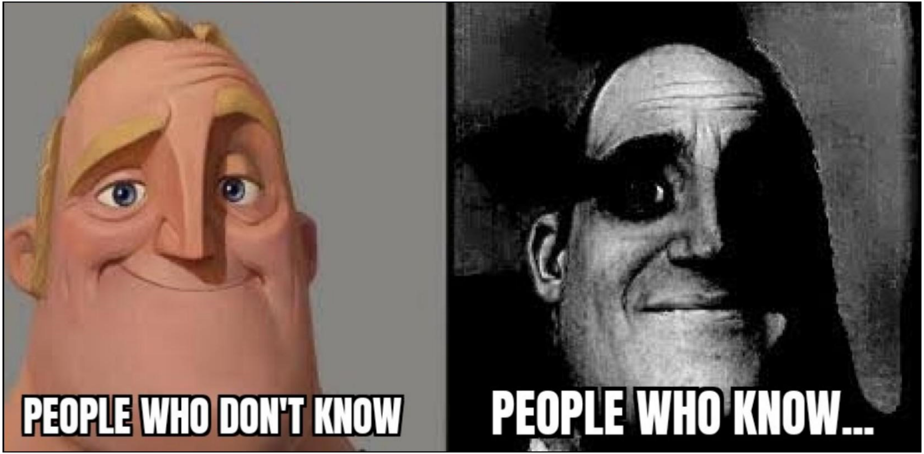
(c) Spreading processes

Some material

- (Ribeiro et al., 2021) survey on algorithms and applications
- (Seshadhri and Tirthapura, 2019) tutorial in WWW 2019 on subgraph counting

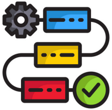
What about temporal motifs?

Temporal Motifs



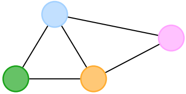
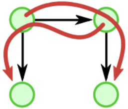
Temporal Motifs

As for temporal communities many definitions exist

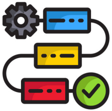


$$\text{Temporal motifs} = \text{static subgraphs} + \text{temporal dynamics} \; (+ \text{additional information})$$

Where

- static subgraph may be (non)induced 
- temporal dynamics time over the static subgraph, in *many ways* 

Temporal Motifs



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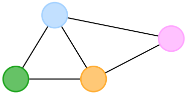
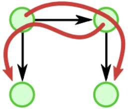
- static subgraph may be (non)induced 
- temporal dynamics time over the static subgraph, in *many ways* 
- additional information is any available *metadata* on nodes or edges, e.g., $f : \{V, E\} \to D$

Let us see some of most used definitions

Temporal Motifs by Kovanen et al. (2011)

Data. $G = (V, E), E = \{(u, v, t)\}$ be a *directed* temporal network

Modeling temporal dynamics

Some definitions

- two edges ($e_1, e_2 \in E$) are Δt-*adjacent* they share at least one node and $|t_{e_1} - t_{e_2}| \leq \Delta t$.

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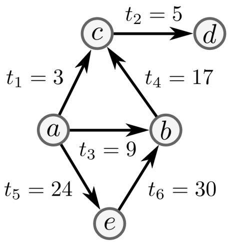
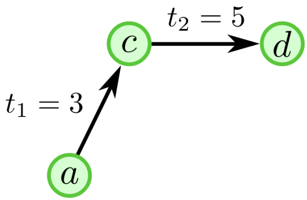
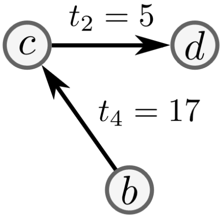
Example. Fix $\Delta t = 10$



(a) Temporal Network G



(b) Δt-adjacent edges



(c) Not Δt-adjacent

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- two edges ($e_1, e_2 \in E$) are Δt-*connected* if there is a sequence Δt-adj. edges from e_1 to e_2 (no time ordering!)

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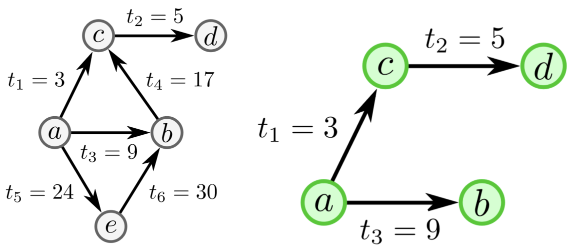
Modeling temporal dynamics

Some definitions

- two edges ($e_1, e_2 \in E$) are Δt-*adjacent* they share at least one node and $|t_{e_1} - t_{e_2}| \leq \Delta t$.
- two edges ($e_1, e_2 \in E$) are Δt-*connected* if there is a sequence Δt-adj. edges from e_1 to e_2 (no time ordering!)

Example. Fix $\Delta t = 10$



(a) Temporal Network G (b) Δt-connected edges

Temporal Motifs by Kovanen et al. (2011)

Data. $G = (V, E), E = \{(u, v, t)\}$ be a *directed* temporal network

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Further, a *connected temporal subgraph* $G' \subseteq G$ constitutes of pairwise Δt-conn. edges

- a conn. temporal subgraph G' is *valid* if for each two events that are incident to a node no other edges are skipped (temporally induced G')

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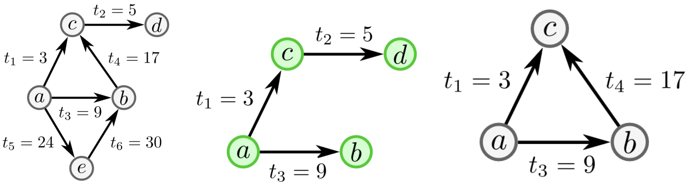

(a) Temporal Network G (b) Valid (c) Not valid

Temporal Motifs by (Kovanen et al., 2011)

Definition

Temporal motifs are *non-isomorphic* classes of subgraphs, where the isomorphism *takes into account edge ordering*.

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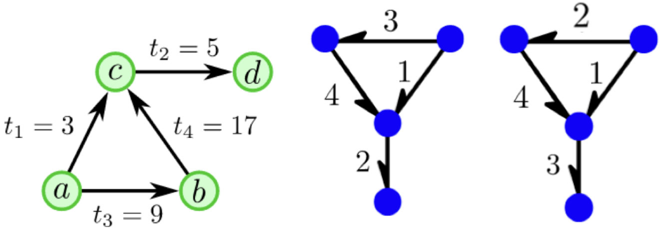

(a) Motif occurrence (b) Class 1? (c) Class 2?

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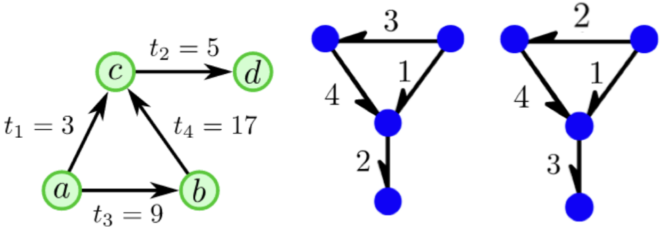

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Temporal Motifs by (Kovanen et al., 2011)

Definition

Temporal motifs are *non-isomorphic* classes of subgraphs, where the isomorphism *takes into account edge ordering*.

Problem

Given G, Δt and a bound k obtain the count of temporal motifs on k nodes.

Solving the counting problem for specific classes

- pre-process G and identify maximal components $\mathcal{O}(|E|)$
- in each component find valid subgraphs of bounded size k, $\mathcal{O}(n^k)$
- map each valid subgraph G' with k-edges on its class (*canonical labeling* is used, exponential in $|G'|$).

A real use case?

Homophily in Phone Call Networks (Kovanen et al., 2013)

Data. Record of 6 months data of mobile phone calls (625million calls) and SMS (207 million). In total >6 million users.

Additional metadata. Sex, age and payment type, combined to obtain 24 different *node colors*

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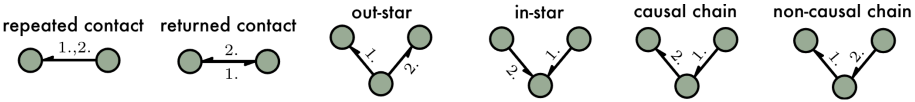
Insights. *Temporal homophily* occurs at gender level, i.e., temporal phone call patterns tend to be different in males and females

Temporal motifs.



The score denotes how strong is data with respect to a random-model. F: female, M: male, $F-F$: all nodes are of F class, $F-*$: there exist at least one node of M class in the motif.

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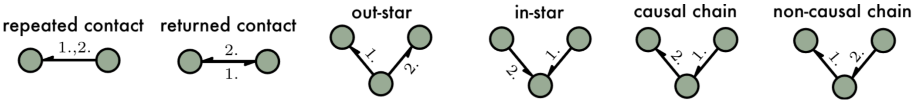

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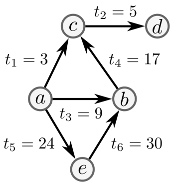
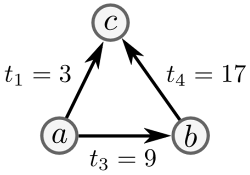
Motif	$F-F$	$F-*$	$M-M$	$M-*$
Repeated calls	1.11	1.11	1.13	1.10
Noncausal chain	**1.08**	**1.02**	1.01	1.04
Causal chain	**1.08**	**1.01**	**0.98**	**1.02**
Out-star	**1.10**	**1.03**	**1.01**	**1.04**

An Issue with Kovanen et al. (2011)'s Definition

Example. Fix $\Delta t = 10$



(a) Temporal Network G



(b) Not valid

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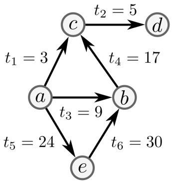

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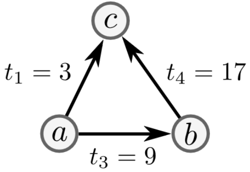


(b) Not valid

Considering only *valid* subgraphs may be **too strict**!

Which can lead to *information loss*, as this can be important for many applications.

How to fix this?

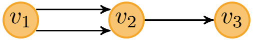
Temporal Motifs by Paranjape et al. (2017)

This model aims at providing a more general and flexible definition of temporal motifs.
Data. $G = (V, E), E = \{(u, v, t)\}$ be a *directed* temporal network

A temporal motif is a pair $M = (K, \sigma)$ (Liu et al., 2019) where

- K is a directed and (weakly)connected multigraph with k-nodes and ℓ-edges.

(a) K_1



(b) K_2

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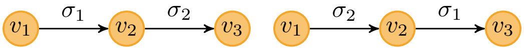
(a) K_1 (b) K_2

- σ is an ordering of the edges of K (modelling temporal dynamics of K)

  Example. Fixing $K = K_1$ then



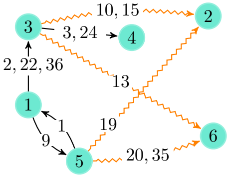
(a) $\sigma_L = \langle (v_1, v_2), (v_2, v_3) \rangle$ (b) $\sigma_R = \langle (v_2, v_3), (v_1, v_2) \rangle$

Note. σ_L is time respecting while σ_R not!

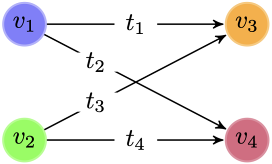
Temporal Motif Counting Problem (Paranjape et al., 2017)

Given G and a value $\delta \in \mathbb{R}^+$, a *time-ordered* sequence $S = \langle (x'_1, y'_1, t'_1), \ldots, (x'_\ell, y'_\ell, t'_\ell) \rangle$ of ℓ unique edges from G is a δ-*instance* of $M = \langle (x_1, y_1), \ldots, (x_\ell, y_\ell) \rangle$ if

1. there exists a bijection h from the vertices appearing in S to the vertices of M, with $h(x'_i) = x_i$ and $h(y'_i) = y_i$, and $i \in [\ell]$;
2. the edges of S occur within δ time; i.e., $t'_\ell - t'_1 \leq \delta$.



(a) Temporal graph G



(b) Temporal motif, t_i gives ordering in σ

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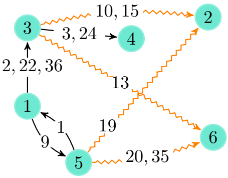

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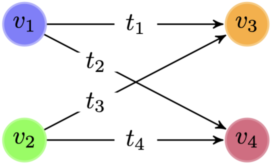

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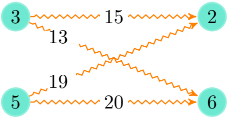
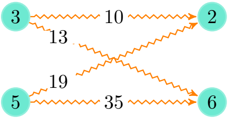
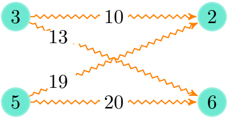
Fix $\delta = 10$



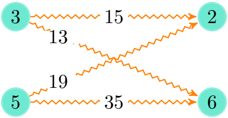
(a) ?



(b) ?



(c) ?



(d) ?

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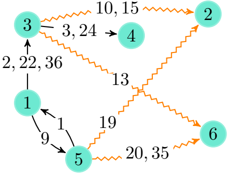

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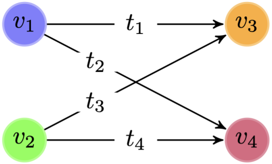

(b) Temporal motif, t_i gives ordering in σ

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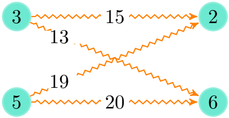
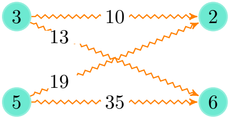
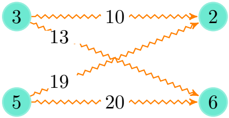


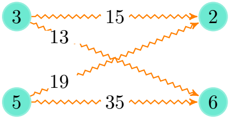
(a) ? ✗



(b) ? ✗



(c) ? ✓



(d) ? ✗

Temporal Motif Counting Problem – cont.

Count of a temporal motif M is: $\#$ of δ-instances of M in G

Problem

Given a temporal network G, a temporal motif M and a parameter $\delta \in \mathbb{R}^+$ obtain *the count* of the temporal motif M

The problem is **NP**-hard

Temporal Motif Counting Problem – cont.

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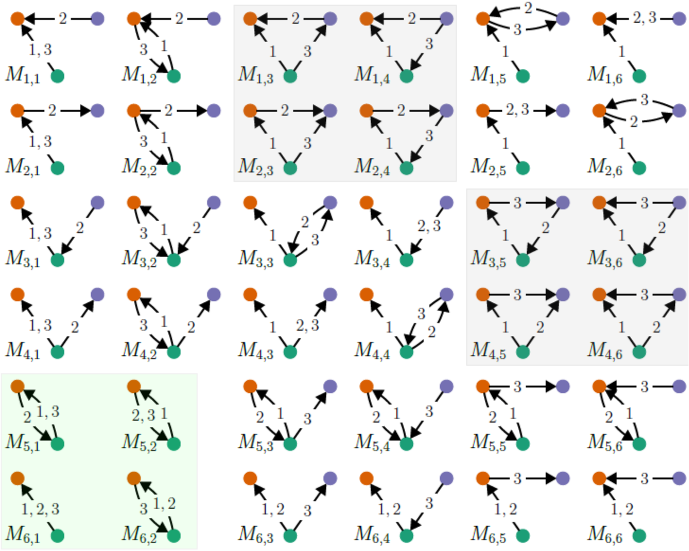
The problem is **NP**-hard even for motifs in **P** for static networks (Liu et al., 2019)!

Lets look at existing algorithms

Exact Algorithm by Paranjape et al. (2017)

The proposed algorithm computes the counts of all $\{2, 3\}$-node 3-edge temporal motifs.



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General framework

- computes the aggregate graph G_A of G
- enumerates all subgraphs $H \subseteq G_A$ isomorphic to K (i.e., $H \sim K$)
- for each H gathers the corresponding temporal networks G_H and sorts edges by timestamps
- applies dynamic-programming to obtain the counts of all the sequences of length ℓ in a window of size δ

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Runtime. $\mathcal{O}(|E| + n^k + \sum_{H \sim K} |E_{G_H}|(\log(|E_{G_H}|) + |E_H|^{\ell})$

Specialized routines for specific motif classes through dynamic programming

Other Exact Algorithms

Other exact approaches

(Mackey et al., 2018) *enumerates* all δ-instances of a fixed temporal motif M without constrains

(Pashanasangi and Seshadhri, 2021) Fast algorithms for temporal triangle counting based on degeneracy ordering

(Gao et al., 2022) improved algorithms for counting $\{2, 3\}$-node 3-edge temporal motifs

(Sarpe, 2023) improved (Mackey et al., 2018) by different matching criteria and timeline partition

(Yuan et al., 2023) dedicated hardware for counting temporal motifs

(Cai et al., 2023) exact algorithms for counting butterflies in temporal bipartite networks

...

As the problem is hard, often better rely on **approximate counting**!

Motif Approximation Problem

Problem

Given a temporal network G, a temporal motif M and a parameter $\delta \in \mathbb{R}^+$, and two additional parameters $\varepsilon, \eta \in (0,1)^2$ obtain C' an estimate of *the count* C of the temporal motif M with

$$\mathbb{P}[|C' - C| \geq \varepsilon C] \leq \eta$$

Why *approximate counts*?

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Given a temporal network G, a temporal motif M and a parameter $\delta \in \mathbb{R}^+$, and two additional parameters $\varepsilon, \eta \in (0,1)^2$ obtain C' an estimate of *the count* C of the temporal motif M with

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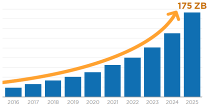
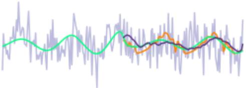
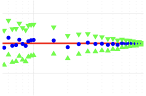
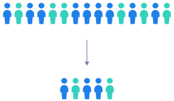
Why *approximate counts*?

- often efficient and practical to compute on massive data 

- approximations are robust to noisy data 
- guarantees on the quality of estimate 

Approximation Algorithms



Most of approximate algorithm are based on randomized sampling

(Liu et al., 2019): partitioning time-span of G in non-overlapping windows and uses importance sampling to decide windows to explore

(Wang et al., 2020): sampling temporal edges with fixed probability, specialized estimators for triangles, and streaming

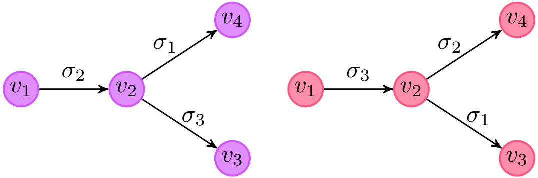
(Sarpe and Vandin, 2021b; Sarpe, 2023): interval based algorithms performing uniform sampling without partitioning

(Pu et al., 2023): edges sampling techniques for counting temporal butterflies on undirected bipartite temporal networks

...

On the Selection of σ (Sarpe and Vandin, 2021a)

A temporal motif is a pair (K, σ), how to properly pick σ?



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Multiple values of σ may need to be tested!

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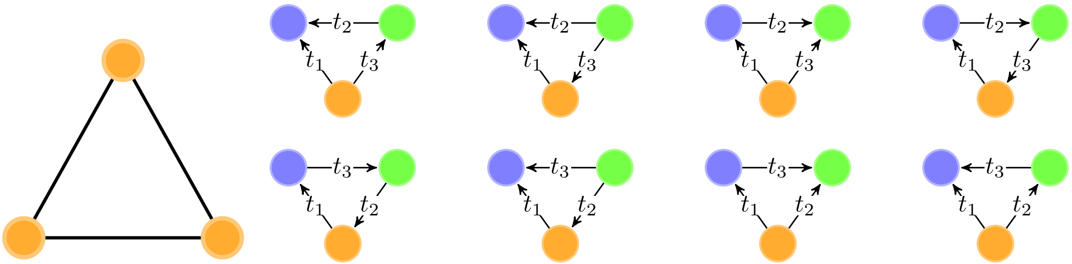
A temporal motif is a pair (K, σ), how to properly pick σ?

Multiple values of σ may need to be tested!

Problem

Given a temporal network G, a parameter $\delta \in \mathbb{R}^+$, a *static* undirected subgraph H and a value $\ell \geq |E_H|$ compute the count of *all* temporal motifs “mapping” on H and having ℓ temporal edges

Example. Fix $\ell = 3$ then

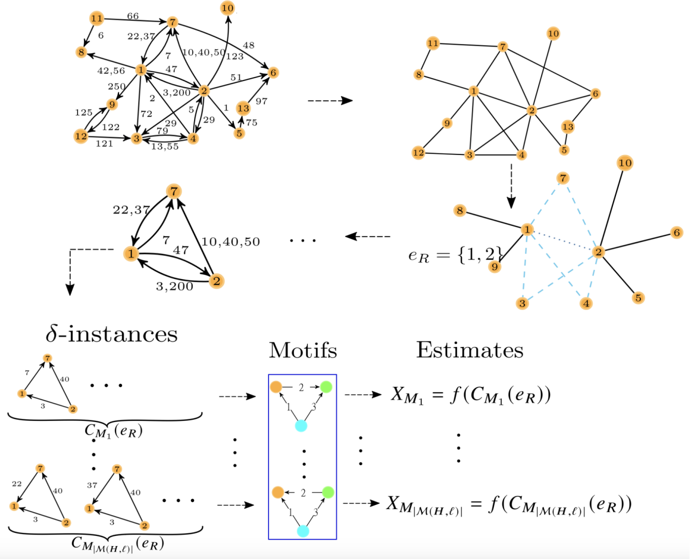


(a) H

(b) motifs to “count”

$\textsc{odeN}$ (Sarpe and Vandin, 2021a)

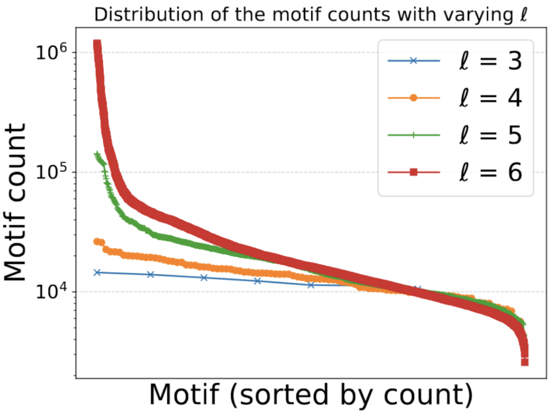
Proposed algorithm $\textsc{odeN}$: randomized sampling algorithm + theoretical guarantees



ODEN (Sarpe and Vandin, 2021a)

Proposed algorithm ODEN: randomized sampling algorithm + theoretical guarantees

Efficiently estimates multiple temporal motifs counts simultaneously. H: triangle and data comes from Facebook posts, varying ℓ.



Are there applications?

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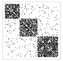
Temporal motifs enabled both novel algorithmic problems and more nuanced applications

Stochastic Block Models (Porter et al., 2022)



Goal. Obtain highly accurate stochastic block models (SBM) to capture temporal motif δ-instances

Proposed solution. Temporal Activity SBM

1. partition nodes according to their activity level {in, out}-edges (resp. C^{in}, c^{out} groups)
2. model temporal edges according

$$\theta = \left(\begin{array}{c|c} \theta_{\mathbf{1},\mathbf{1}} & \theta_{\mathbf{1},\mathbf{2}} \\ \hline \theta_{\mathbf{2},\mathbf{1}} & \theta_{\mathbf{2},\mathbf{2}} \end{array}\right)$$

   where $\theta \in \mathbb{R}^{C^{in} \times C^{out}}$ models edge occurrence

3. analytical computation of motif counts according to such model

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Goal. Obtain highly accurate stochastic block models (SBM) to capture temporal motif δ-instances

Proposed solution. Temporal Activity SBM

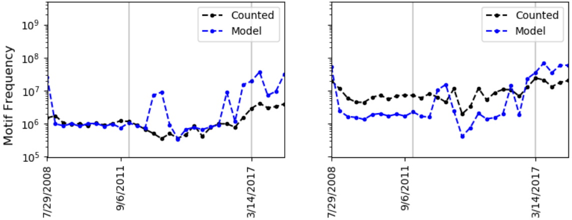
1. partition nodes according to their activity level {in, out}-edges (resp. C^{in}, c^{out} groups)
2. model temporal edges according

$$\theta = \left(\begin{array}{c|c} \boldsymbol{\theta_{1,1}} & \boldsymbol{\theta_{1,2}} \\ \hline \boldsymbol{\theta_{2,1}} & \boldsymbol{\theta_{2,2}} \end{array}\right)$$

   where $\theta \in \mathbb{R}^{C^{in} \times C^{out}}$ models edge occurrence

3. analytical computation of motif counts according to such model

The **model accurately tracks temporal motif counts**. Financial dataset recorded over 10 years, $\delta = 90$ days, left $M_1 = \langle (v_1, v_2), (v_3, v_2), (v_1, v_2) \rangle$, right $M_2 = \langle (v_1, v_2), (v_2, v_1), (v_2, v_1) \rangle$



Synthetic Network Generators (Liu and Sariyüce, 2023)

Goal. Obtain a synthetic temporal network, similar to the one in input for temporal motifs

Proposed solution. Motif Transition Model

Cold event (CE): first event on a temporal motif instance

1. compute temporal network statistics
   - static degree distribution (K_{CE}) and timestamps (T_{CE}) of cold events
   - $\mathcal{P}$ motif-transition properties (how likely are motifs to evolve from one to another)
   - Λ motif transition rated (how often they transition)
   - μ number of static edges involved in transitions.
2. generate static network from K_{CE} and assign (T_{CE})
3. simulate transition process according to computed metrics

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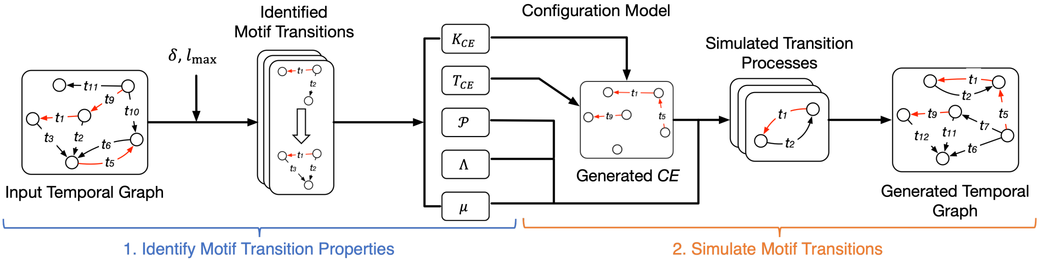
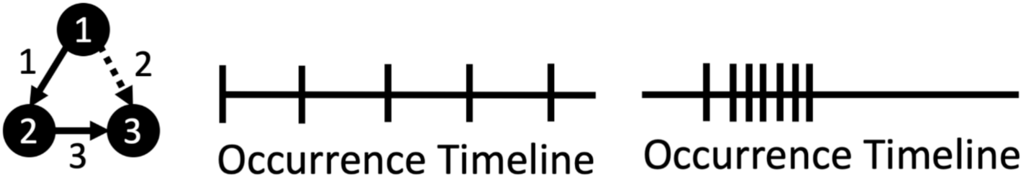


Mining Persistent Events (Belth et al., 2020)

Goal. Distinguish between how motif occur over time *streams*



Mining Persistent Events (Belth et al., 2020)

Goal. Distinguish between how motif occur over time *streams*

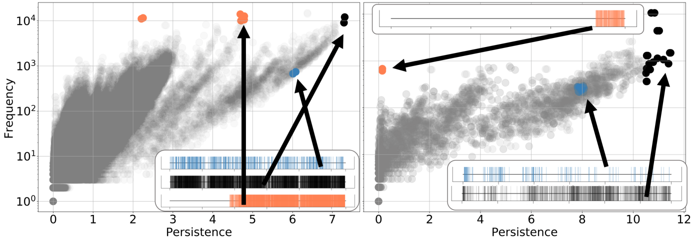
Proposed solution. Assign persistence score and algorithms to compute it

Let x be an event (e.g., temporal motif instance) then the persistence $P(\cdot)$

$$P(x) = f\left(\underbrace{W(x)}_{\text{width}}, \underbrace{F(x)}_{\text{frequency}}, \underbrace{S(x)}_{\text{uniformity}}\right)$$

Online and offline streaming algorithms are developed, efficient for small size of events (small ℓ)

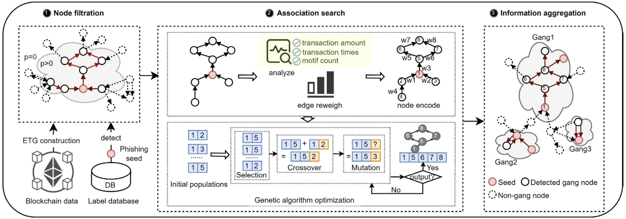
The score allows to distinguish between frequent/infrequence and bursty/persistent

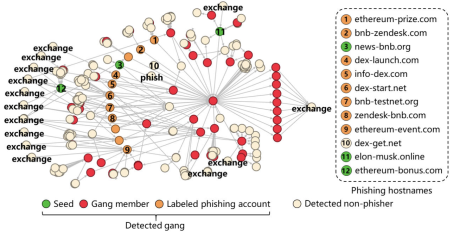


(a) Boston Bike **(b)** Columbus Bike

Some Practical Applications (Liu et al., 2024)

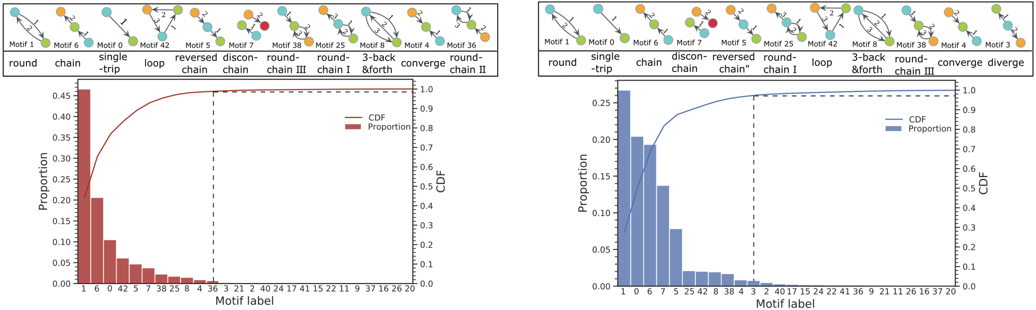
- capturing high-order patterns for phishing gang identification on cryptocurrency networks





Some Practical Applications (Lei et al., 2020)

- analyzing different temporal travel patterns in people commuting (metro vs bike sharing)

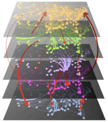


(a) using metro (b) using bikesharing

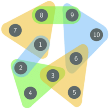
Other Temporal Motif Definitions

Several other definitions exist in literature

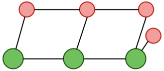
- (Boekhout et al., 2019): studied temporal multilayer motifs



- (Lee and Shin, 2023): studied temporal hypergraph motifs

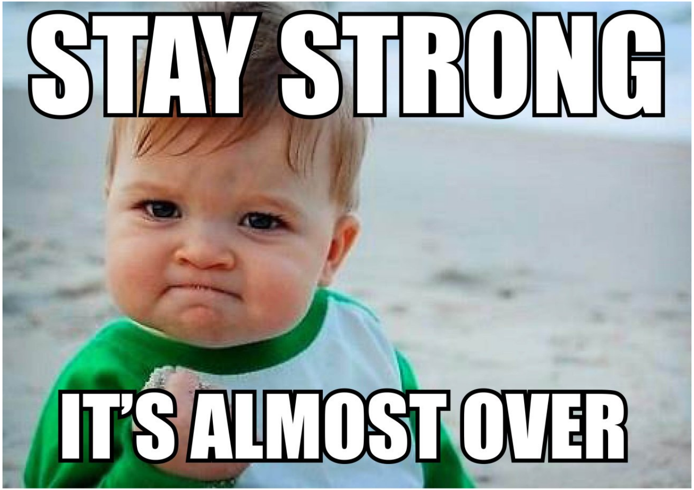


- (Longa et al., 2021): studied motifs based ego-networks



- (Kosyfaki et al., 2018): defined motifs for temporal networks with flows
- ...

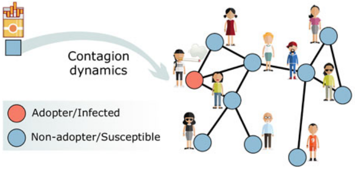
If you want to know more, check the survey by Liu et al. (2021)


STAY STRONG
IT'S ALMOST OVER

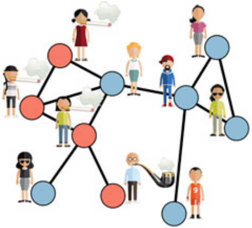
Diffusion and Random Networks

Diffusion Analysis and Spreading

- propagation models
  - used to study disease spreading or information cascade in the network
- activity spreading: virus, information, idea, rumor
- applications: epidemiology, information security, marketing
- why use models?
  - facilitate mathematical analysis of propagation processes
  - have intuitive interpretation
  - proven to be realistic by empirical studies
- extensive survey in the book (Shakarian et al., 2015)



(a) $t = 1$



(b) $t = 2$

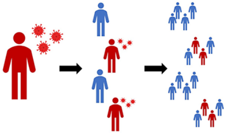
Standard Models

Most used models are

- susceptible-infected (SI) model
  - SIR, SIRS, other variants
- independent cascade (IC) model
- linear threshold (LT) model

Such models are *important building blocks* for many data mining formulations!

Susceptible-Infectious (SI) Model

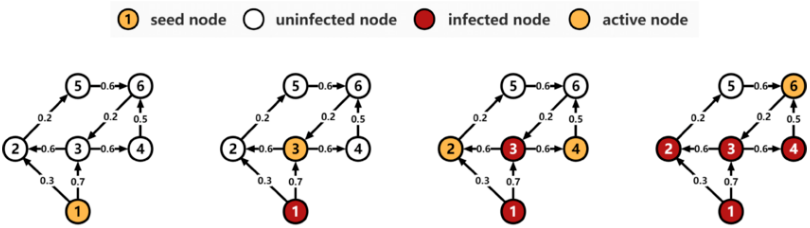


- beginning
  - time step t_0
  - one or several infected nodes in I_{t_0} (seeds of infection)
- subsequent timestamp t
  - all infected nodes try to infect each of their *susceptible* neighbors
  - with probability p infection is *passed through an edge*
  - if a node receives infection becomes infected
- the process continues until all nodes are infected.

Some *other node types*, recovered (nodes that were infectious and now cannot spread), and exposed (infected that cannot spread)

Independent Cascade (IC) Model

- nodes can be in either susceptible or infectious
- each edge (u, v) has an *individual* infection probability (based on proximity, frequency, etc..)
- infected node u has a *single* chance to infect its neighbors



(a) Time step 0 (b) Time step 1 (c) Time step 2 (d) Time step 3

Used to study new propagation of ideas, concepts, or products (Kempe et al., 2003; Wang et al., 2012)

Linear Threshold (LT) Model

- every edge (u, v) has a probability $p(u, v)$
- at the time step t, u: susceptible $\to$ infectious, if the total weight from its infectious neighbors is larger than a random propagation threshold θ_u

$$\sum_{v \in N(u)} p(v, u)\mathbf{1}[v \text{ is infectious at } t] \geq \theta_u$$

- conditional on thresholds and the initially infected nodes the process is *deterministic*.

LT model has applications in viral-marketing (Chen et al., 2010; Goyal et al., 2011)

Mining Applications

Powerful modeling for many **mining primitives**

1. **immunization strategies**, e.g., find smallest set of nodes to stop a spreading process. (Lee et al., 2012; Yu et al., 2010; Starnini et al., 2013; Génois et al., 2015; Mantzaris and Higham, 2016; Valdano et al., 2015; Gauvin et al., 2015)
2. **influence maximization**, e.g., select the initial set of seeds, to optimize diffusion, applications in marketing and network design. (Aggarwal et al., 2012; Zhuang et al., 2013; Gayraud et al., 2015; Rodriguez et al., 2011; Gomez-Rodriguez et al., 2016; Chen et al., 2012; Liu et al., 2012; Rodriguez and Schölkopf, 2012; Du et al., 2013)
3. **seed and cascade reconstruction**, e.g., given some observed data of a spreading phenomenon, find the *most probable seed nodes* or *cascades*, applications in epidemiology and nfluencer discovery. (Shah and Zaman, 2011; Lappas et al., 2010; Prakash et al., 2012)

Random Models

Common questions in temporal data analysis

- *how novel* is this result?
- is this only due to *random chance*?
- are there *properties in the data explaining* the results?

To find an **answer** $\rightarrow$ use a **statistical test** (Pellegrina et al., 2019)

Random Models

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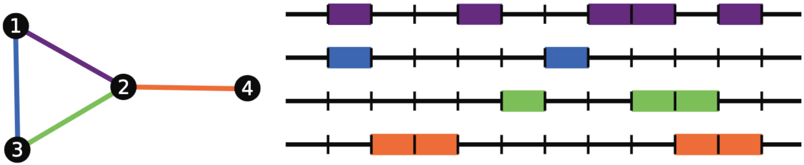
- are there *properties in the data explaining* the results?

To find an **answer** $\rightarrow$ use a **statistical test** (Pellegrina et al., 2019)

- start from a **temporal network** G
- formulate an **hypothesis** (H_0) about data (e.g., time does not matter for $f(G)$)
- perform a **test to reject** H_0, usually
  - generate multiple datasets $G_1^{H_0}, \ldots, G_L^{H_0}$ for some large L according to H_0
  - compute some function $g(G_1^{H_0}, \ldots, G_L^{H_0})$ to reject H_0 (e.g., $g(\cdot)$ not explains $f(G)$).

Random Models Gauvin et al. (2022)

Randomized models are used to test temporal/static properties in data
A temporal network as time-line of events



We have

1. static structure (SS)
2. timeline associated to its links (TL)

To obtain random models → use *these two properties or combinations* of the two

Random Models Gauvin et al. (2022)

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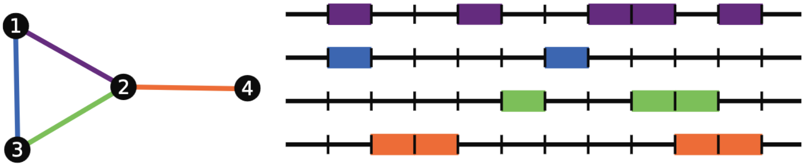
A temporal network as time-line of events



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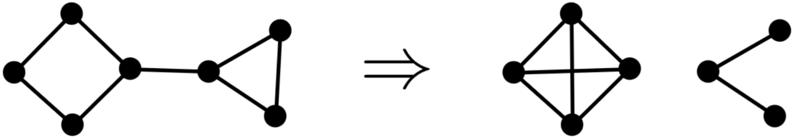
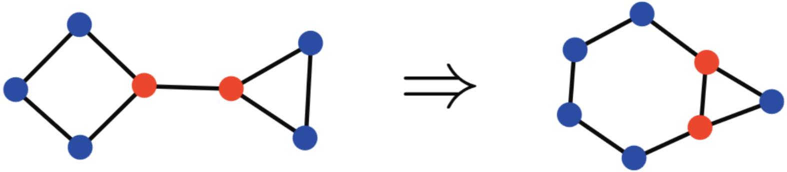
Let us see some examples

Random Models Gauvin et al. (2022)

Shuffling *only* static properties while fixing the temporal ones



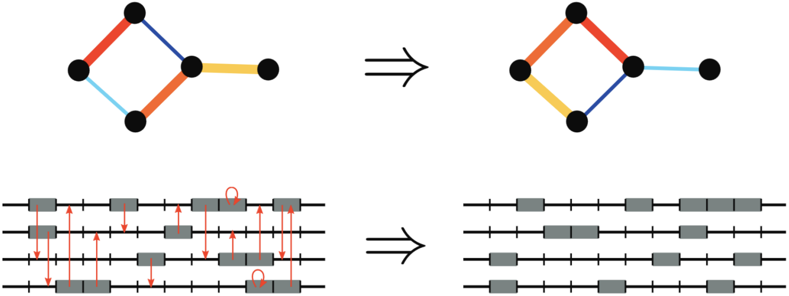
(a) *link shuffling*



(b) *(conn.) constrained degree link shuffling*

Random Models Gauvin et al. (2022)

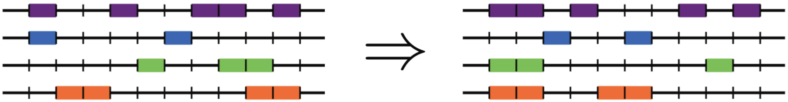
timeline shuffling



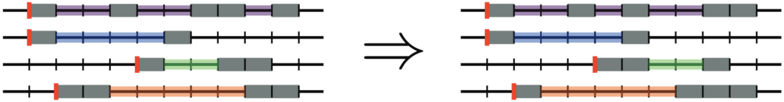
This model retains static properties and conditions on the observed temporal ones

Random Models Gauvin et al. (2022)

Other models retaining static properties



(a) *shuffling events over each timeline*

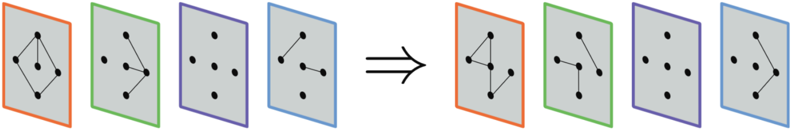


(b) *shuffling events and retaining gaps*

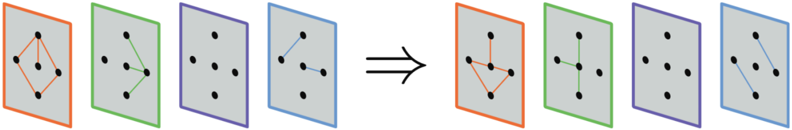
... and much more such combinations (static + temporal) ...

Random Models Gauvin et al. (2022)

Some random models for snapshot-based temporal networks



(a) *snapshot shuffling*



(b) *isomorphism based*

Random Models

Summary.

- *random models* can be of fundamental importance for testing significance/generating additional data
- they can be applied for most of the mining problems that we discussed
- some of them may be *hard to compute* and **new methods may be required**

Agenda

Part I : **Introduction and Motivation**

- models of temporal networks
- algorithmic approaches

Part II : **Mining Temporal Networks A:**

- connectivity, temporal properties
- centrality, cores

Part III : **Mining Temporal Networks B:**

- communities, patterns and events
- diffusion and random networks

Part IV : **Tools and Code Libraries**

Part V : **Challenges, Open Problems, and Trends**

Part IV

Tools and Code Libraries

Tools Overview

Tool Name	Language	Functionalities
SNAP	C++/Python	Temporal motifs
Graph-tool	Python/C++	Simulate network dynamics (e.g., spreading)
Teneto	Python	Temporal network measures (centrality, reachability, etc..), community detection, visualization
Phasik	Python	Infer temporal networks from time series data
Reticula	C++/Python	Random networks, random models, temporal reachability, events...
Tglib	C++/Python	Paths, centrality and other properties (cores, clustering coefficient, etc..)
Raphtory	Rust/Python	Centrality, communities, cores, motifs, null models, visualization and more...

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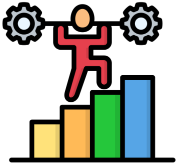
Part V

Challenges, Open Problems, and Trends

Challenges in Temporal Network Mining



- large number of **problem formulations** and variants
- gaps **fundamental theoretical** treatment
  - many are combinations of several ideas of static cases
  - require often many parameters
- hard to compare methods and **choose based on applications**
  - few datasets with ground-truth solutions
  - synthetic generators are built on various assumptions
  - no standards and benchmarks
  - as always: lack of useful and rich datasets
- a large number of quality metrics to calculate and compare
- comparisons are misleading if methods are designed for other definitions

Directions in Temporal Network Mining

- **more systematic** approaches, **quality guarantees**
- **interpretability** of the results
- **diversity** and **fairness**

- applications and application-tailored algorithms
  - encourage *interdisciplinary research* and collaborations
  - computational *social science*

Thanks for your attention!

https://miningtemporalnetworks.github.io/

menti.com

code: 14 46 97 6

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